TSE Master 2 — Macroeconomics I
Problem Set 1

Yue FEI and Lan LAN *

Here is some fundamentals of Balanced Growth Path (BGP)

Growth rate under Balanced Growth Path (BGP)

Corollary

We have $A_t \geq 0$, $B_t \geq 0$, $C_t \geq 0$ for $\forall t$, and

$$A_t + B_t = C_t \tag{1}$$

Then if there exists BGP, we must have that A_t, B_t, C_t grow at the same rate along BGP.

proof: Let $\frac{\dot{A}_t}{A_t} = g_A$, $\frac{\dot{B}_t}{B_t} = g_B$, $\frac{\dot{C}_t}{C_t} = g_C$. From (1), we know that for $\forall t$,

$$g_A \frac{A_t}{C_t} + g_B \frac{B_t}{C_t} = g_C \tag{2}$$

Let us denote $\frac{A_t}{C_t} = x_t^A$, $\frac{B_t}{C_t} = x_t^B$. Then (2) can be rewritten as:

$$g_A x_t^A + g_B x_t^B = g_C \tag{3}$$

and (1) can be rewritten as:

$$x_t^A + x_t^B = 1 \tag{4}$$

Differentiating (3) and (4) on t, yields:

$$g_A \cdot \dot{x_t^A} + g_B \cdot \dot{x_t^B} = 0$$

$$\dot{x_t^A} + \dot{y_t^B} = 0$$

This implies that:

$$\dot{x_t^A} = 0, \dot{x_t^B} = 0$$

*Thanks to Alban for providing solutions from previous years. Please let us know if you find typos or errors: fei.yue@hotmail.com or lan.lansl@yahoo.fr

or

$$g_A = g_B$$

$$\dot{x}_t^A + \dot{x}_t^B = 0$$

If the former is true, then we have $g_A = g_C = g_B$, if the latter is true, we plug this into (2), we also have $g_A = g_B = g_C$

1 Growth with investment-specific technological change

1. There are two ways to understand the resource constraint.

The first way. Let $X_t I_t = \mathbb{I}_t$. Rewriting the resource constraint and Law of Motion for capital as:

$$Y_t = C_t + \mathbb{I}_t$$

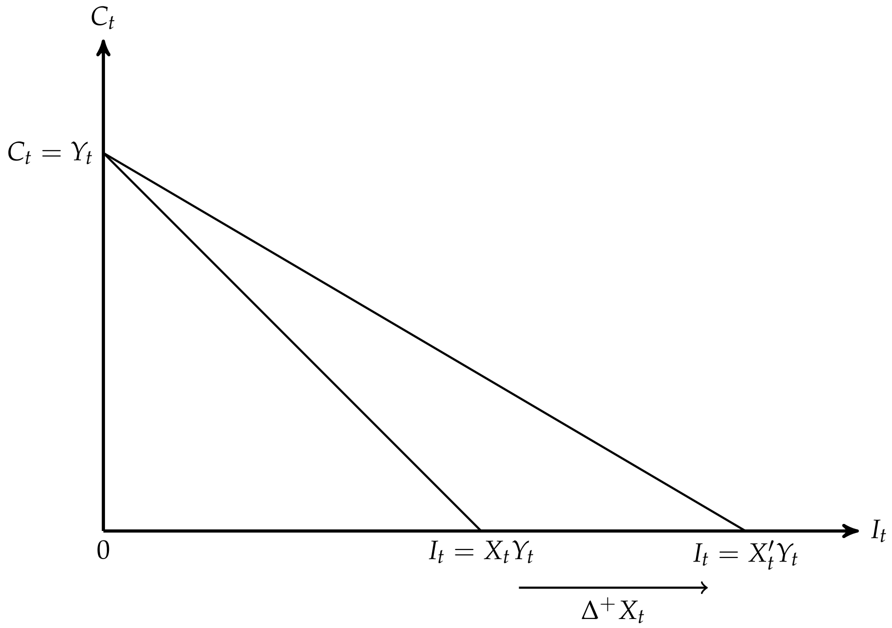
$$\dot{K}_t = X_t \cdot \mathbb{I} - \delta K_t$$

These two formulas mean that: of all the capital stock available, a fraction of X_t is actually put into use. This can be seen as the efficiency of capital usage.

The second way. Rewriting the equation in the (I_t, C_t) plane yields

$$C_t = Y_t - \frac{I_t}{X_t},$$

showing that Y_t affects the level of the production frontier while X_t controls its slope, as illustrated in the figure below. In particular, for any decomposition of Y_t into goods devoted to consumption and goods devoted to investment, a larger value of X_t means that a higher I_t is feasible, while C_t is unaffected. In that sense, X_t can be interpreted as investment-specific technological progress.



2. Using the resource constraint to write $I_t = X_t(Y_t - C_t)$, the planning problem is

$$\max_{C_t, K_t} \int_0^\infty e^{-\rho t} \ln C_t dt \tag{5}$$

subject to

$$\dot{K}_t = X_t(A_t K_t^\alpha - C_t) - \delta K_t. \tag{6}$$

The Hamilton-Jacobian-Bellman equation is written as:

$$\rho V = \max_{C_t} u(C_t) + V'(K_t)\dot{K}_t$$

We take differentiation of the Bellman equation w.r.t. both C_t and K_t yields:

$$\begin{aligned}
0 &= \frac{1}{C_t} - V'(K_t)X_t \\
\rho V_{K_t} &= \frac{\partial V' K_t \dot{K}_t}{\partial K_t} \\
&= V'(K_t) \cdot \frac{\dot{K}_t}{K_t} + \frac{\partial V'(K_t)}{\partial K_t} \cdot \frac{K_t}{t} \\
&= V'(K_t) \cdot \frac{d\dot{K}_t}{dK_t} + \frac{dV' K_t}{dt}
\end{aligned}$$

Rearranging V_{K_t} yields that:

$$\rho = \frac{\dot{K}_t}{K_t} + \frac{V'(\dot{K_t})}{V'(K_t)}$$

Then by plugging the Law of Motion of capital, we obtain:

$$\rho = \alpha X_t A_t K_t^{\alpha-1} + \frac{V'(\dot{K_t})}{V'(K_t)}$$

This equals to:

$$\frac{V'(\dot{K_t})}{V'(K_t)} = \rho - \alpha A_t K_t^{\alpha-1} + \delta$$

From the F.O.C of consumption, we have,

$$\begin{aligned}
\frac{\dot{C}_t}{C_t} &= -(\frac{V'(\dot{K_t})}{V'(K_t)} + \frac{\dot{X}_t}{X_t} \\
&= \alpha A_t X_t K_t^{\alpha-1} - \delta - \rho - g_x
\end{aligned}$$

This is the continuous time Euler equation.

3. We now compute the balanced growth path of the economy. Let g_Y, g_C, g_I, and g_K denote the constant growth rates of output, consumption, investment, and the capital stock along the BGP. To solve for these variables, we use the restrictions implied by the model: the production function

$Y_t = A_t K_t^\alpha$ implies

$$g_Y = g_A + \alpha g_K,$$

the capital accumulation equation $dK_t/dt = I_t - \delta K_t$ implies

$$g_I = g_K,$$

the economy's resource constraint $Y_t = C_t + I_t/X_t$ implies

$$g_Y = g_C = g_I - g_X,$$

and the Euler equation for consumption implies

$$g_X + g_A = (1-\alpha)g_K.$$

It follows that

$$g_I = g_K = \frac{g_A + g_X}{1-\alpha},$$

while

$$g_Y = g_C = \frac{g_A + \alpha g_X}{1-\alpha}.$$

Since $\alpha \in (0,1)$, notice that along the balanced growth path, investment and the capital stock grow faster than output and consumption. We return to this property below.

4. The complete market implies that market equilibrium is the same as social planner outcome. In this problem, there are two markets. One final goods market, and one capital goods market. The market clearing condition for the two markets can be seen as below:

$$P_t^C C_t + P_t^I S_t = P_t^Y Y_t$$

$$S_t = I_t$$

Here S_t stands for the saving of households. Combine these two conditions, we have the Market Clearing condition (MCC) for the whole economy :

$$P_t^C C_t + P_t^I I_t = P_t^Y Y_t$$

Now, we prove that if the economy satisfies the resource constraint, then we must have $\frac{P_t^I}{P_t^C} = \frac{1}{X_t}$ let $C_t = 0$, then

$$\frac{P_t^I}{P_t^Y} = \frac{1}{X_t}$$

let $I_t = 0$, then

$$\frac{P_t^C}{P_t^Y} = 1$$

therefore, we have the price ratio equals to the inverse of X_t. (Note that you do not solve the market equilibrium, since there is no maximization involved.)

Gordon's (1990) observation that P^I/P^C has been falling throughout the postwar period in the U.S. then implies that X_t has been growing, so that $g_X > 0$. Therefore, at least some part of the recent U.S. economic growth has been due to investment-specific technological progress. Greenwood, Hercowitz, and Krusell (1997) argue that this contribution represents as much as 60% of growth in output per hour worked in the postwar period in the U.S.

5. In question 3., we have shown that $g_K = g_I = (g_A + g_X)/(1-\alpha) > g_Y = (g_A + \alpha g_X)/(1-\alpha)$, so that the capital-output ratio K_t/Y_t and the investment-output ratio I_t/Y_t are growing at rate g_X along the balanced growth path. Therefore, the model seems to violate some of the Kaldor facts.

To reconcile the model with the data, notice that, unlike in the one-sector growth model, K_t and Y_t do not have the same units here. Instead of considering the capital-output ratio, it thus seems more relevant to compute the relative value of the capital stock to output, given by

$$\frac{P_t^I K_t}{P_t^C Y_t} = \frac{K_t/X_t}{Y_t},$$

whose growth rate along the BGP is equal to $g_K - g_X - g_Y = 0$. Likewise, $(P_t^I I_t)/(P_t^C Y_t)$ is constant along the BGP. Therefore, even though physical capital and investment grow faster than output, their values in terms of output remain constant and the model is consistent with a broader interpretation of the Kaldor facts.

2 The open-economy Ramsey model with installation costs for capital

This problem extends the textbook Ramsey model to the open economy and introduces costs to installing capital. Allowing international trade in goods and assets modifies the resource constraint of the economy, through foreign borrowing or lending. Also, capital adjustment costs impact investment decisions. A useful reference for this exercise is Blanchard and Fischer (1989, section 2.4).

1. At each date t, the surplus of the trade balance is the difference between local resources (output) and local needs (consumption, investment, and installation costs):

$$TB_t = AK_t^\alpha - C_t - I_t - f(I_t),$$

where recall that $f(I)$ is the cost of installing I units of capital, expressed in terms of the final good.

2. Define W_t the net stock of foreign assets owned by the economy at date t, which increases with

interest payments from the rest of the world and with the current trade balance:

$$\begin{aligned}\frac{dW_t}{dt} &= rW_t + TB_t \\ &= rW_t + AK_t^\alpha - C_t - I_t - f(I_t),\end{aligned} \tag{7}$$

where r is the (constant) world interest rate. Notice that if $W_t > 0$, the economy is a net lender and $rW_t > 0$ represents interest payments from the rest of the world, while if $W_t < 0$ the economy is a net borrower and $rW_t < 0$ represents interest payments to the rest of the world.

3. Equation (7) is an ordinary differential equation in W_t, which can be solved by integration between 0 and $+\infty$. We have

$$\frac{dW_t}{dt} = rW_t + AK_t^\alpha - C_t - I_t - f(I_t)$$

$$\begin{aligned}&\iff & e^{-rt}\frac{dW_t}{dt} - re^{-rt}W_t &= e^{-rt}[AK_t^\alpha - C_t - I_t - f(I_t)] \\ &\iff & \frac{d}{dt}(e^{-rt}W_t) &= e^{-rt}[AK_t^\alpha - C_t - I_t - f(I_t)] \\ &\implies & \lim_{t\to\infty} e^{-rt}W_t - W_0 &= \int_0^\infty e^{-rt}[AK_t^\alpha - C_t - I_t - f(I_t)]dt.\end{aligned}$$

Imposing the solvency constraint $\lim_{t\to\infty} e^{-rt}W_t \geq 0$, we get

$$W_0 + \int_0^\infty e^{-rt}[AK_t^\alpha - C_t - I_t - f(I_t)]dt \geq 0, \tag{8}$$

which is the country's intertemporal budget constraint. It states that the discounted sum of expenditures cannot exceed the initial asset position plus the discounted sum of resources.

4. Rearranging the intertemporal budget constraint (8) yields

$$\int_0^\infty e^{-rt}C_t dt \leq W_0 + \int_0^\infty e^{-rt}[AK_t^\alpha - I_t - f(I_t)]dt. \tag{9}$$

This constraint bounds the set of possible trajectories for consumption. Moreover, since the representative consumer's utility function only depends on consumption, it is clear that the optimal investment strategy must maximize the right-hand side of (9) with respect to I_t to expand the set of feasible consumption trajectories (recall monotonicity of preferences). Thus, the optimal investment strategy solves

$$\begin{aligned}\max_{I_t,K_t} \quad & \int_0^\infty e^{-rt}[AK_t^\alpha - I_t - f(I_t)]dt \\ \text{s.t.} \quad & \frac{dK_t}{dt} = I_t, \\ & K_t \geq 0.\end{aligned}$$

5. We solve the problem using optimal control techniques (see Acemoglu's textbook (2009, Chapter 7) for an excellent presentation). For simplicity, we neglect the constraint $K_t \geq 0$ and we assume the existence of an interior solution. Here, the state variable is K_t and the control variable

is I_t. Letting λ_t denote the costate variable, the Hamiltonian of the problem writes

$$H(K_t, I_t, \lambda_t, t) = e^{-rt}[AK_t^\alpha - I_t - f(I_t)] + \lambda_t I_t.$$

First-order optimality conditions are

$$\begin{aligned} \frac{\partial H}{\partial I_t} = 0 \quad &\iff \quad e^{-rt}[-1 - f'(I_t)] + \lambda_t = 0 \\ \frac{\partial H}{\partial K_t} = -\frac{d\lambda_t}{dt} \quad &\iff \quad e^{-rt}\alpha A K_t^{\alpha-1} = -\frac{d\lambda_t}{dt} \\ \lim_{t\to\infty} \lambda_t K_t = 0. & \end{aligned}$$

The last condition is the so-called transversality condition. Introducing a new variable $q_t = e^{rt}\lambda_t$, the above system rewrites

$$q_t = 1 + f'(I_t), \tag{10}$$

$$\frac{dq_t}{dt} = rq_t - \alpha A K_t^{\alpha-1}, \tag{11}$$

$$\lim_{t\to\infty} e^{-rt} q_t K_t = 0. \tag{12}$$

Moreover, by assumption f is such that $f'' > 0$, so that f' is a bijection from $\mathbb{R}$ to $f'(\mathbb{R})$. It can thus be inverted, and equation (10) gives

$$I_t = g(q_t) = (f')^{-1}(q_t - 1).$$

From the properties of f', we easily get that g is strictly increasing (since $f'' > 0$) and that $g(1) = (f')^{-1}(0) = 0$ (since $f'(0) = 0$), as was to be shown.

6. Since q_t is the costate variable in the current-value Hamiltonian (see the appendix), it can be interpreted as the current shadow value of capital. From (10), it is also equal to the marginal cost of investment. This equality is intuitive: at the optimum, the agent equates the marginal cost of investing to the marginal value of the capital stock.

We can get one more interpretation of q_t. Multiplying (11) by e^{-rt}, we obtain

$$e^{-rt}\frac{dq_t}{dt} - re^{-rt}q_t = -e^{-rt}\alpha A K_t^{\alpha-1} \qquad \iff \qquad \frac{d}{dt}(-e^{-rt}q_t) = e^{-rt}\alpha A K_t^{\alpha-1}.$$

Integrating forward, we get

$$-\lim_{T\to\infty} e^{-rT}q_T + e^{-rt}q_t = \int_t^\infty e^{-r\tau}\alpha A K_\tau^{\alpha-1} d\tau. \tag{13}$$

To get rid of the limit, we use the household's optimality conditions. The transversality condition imposes that

$$\lim_{t\to\infty} e^{-rt} q_t K_t = 0.$$

Notice that if $\lim_{t\to\infty} K_t = 0$, we have a contradiction: equation (11) implies that $q_t \to -\infty$, but this case can be ruled out (see the appendix). Hence, to satisfy the transversality condition it must be that $\lim_{t\to\infty} e^{-rt}q_t = 0$. Equation (13) thus becomes

$$e^{-rt}q_t = \int_t^\infty e^{-r\tau}\alpha A K_\tau^{\alpha-1} d\tau \quad \Longleftrightarrow \quad q_t = \int_t^\infty e^{-r(\tau-t)}\alpha A K_\tau^{\alpha-1} d\tau. \tag{14}$$

Thus, q_t is also the discounted value at date t of all future marginal products of capital. Notice that this equation implies that $q_t > 0$.

7. At the steady state, both the capital stock and investment are constant. Since $dK_t/dt = I_t$, it must be that $I^* = 0$. It then follows from equation (10) that $q^* = 1$, and from equation (11) that $\alpha A (K^*)^{\alpha-1} = r$. The corresponding capital stock is thus

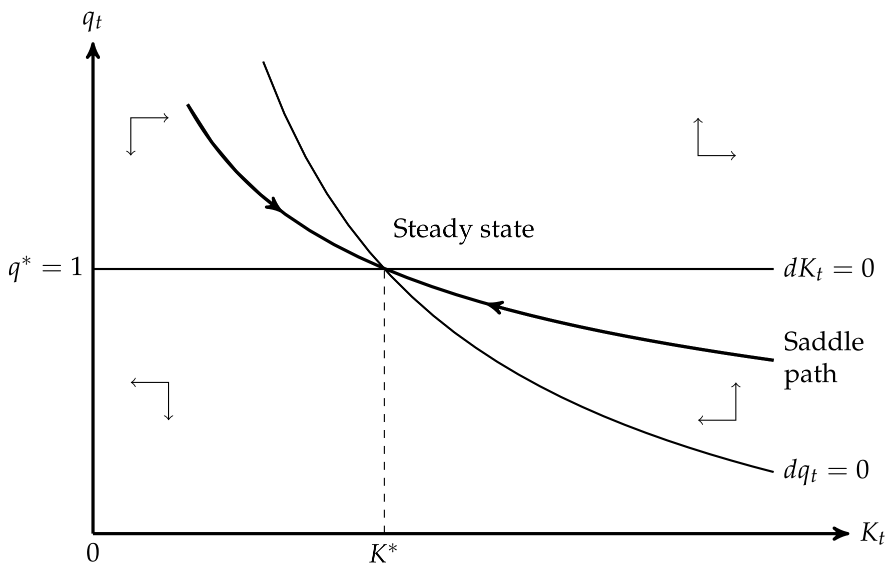
$$K^* = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}.$$

The steady-state marginal product of capital is equal to the world interest rate. This is an arbitrage condition from the point of view of investors: for example, if the rate of return of investing in world capital markets were larger than that of investing in the domestic capital stock, no agent would choose to hold K_t and markets would not clear.

8. We now draw the phase diagram of the economy in the (K_t, q_t) space. The relevant laws of motions are

$$\frac{dq_t}{dt} = rq_t - \alpha A K_t^{\alpha-1},$$
$$\frac{dK_t}{dt} = g(q_t).$$

The locus $\{dq_t/dt = 0\}$ is described by the relation $rq_t = \alpha A K_t^{\alpha-1}$, which is decreasing and convex in K_t. The locus $\{dK_t/dt = 0\}$ is described by the restriction $q_t = 1$, which is just a horizontal line. Hence, the phase diagram is

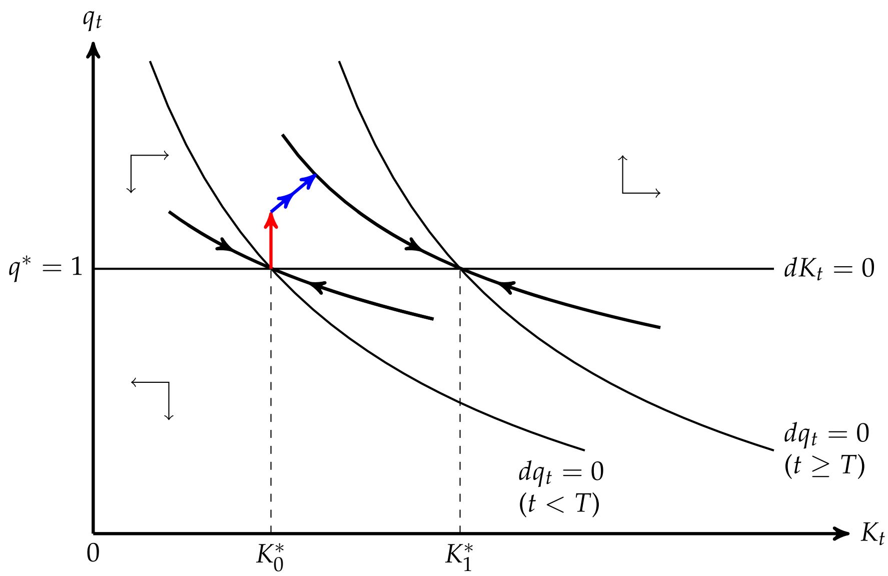


The saddle path is the only stable trajectory: it is the one characterized by the optimality conditions (10)-(12). There are various ways to rule out other trajectories. The most rigorous one is to apply Theorem 7.14 in Acemoglu (2009) as done in the appendix, which proves that the optimality conditions derived from the Hamiltonian are both necessary and sufficient. Since these conditions have only one solution, the optimal trajectory must be unique.

Alternatively, it is possible to rule out trajectories that do not converge to the steady state by showing that they violate the optimality conditions. For example, if the economy starts above the saddle path, then $q_t \to \infty$, $K_t \to \infty$, and $dq_t/dt \to rq_t$, so that $q_t \approx e^{rt}$ and the transversality condition is violated. If the economy starts below the saddle path, then $q_t \to 0$ and $K_t \to 0$. Equation (11) then implies that $q_t \to -\infty$, which violates the restriction that $q_t > 0$.

9. Assume that the economy is at its initial steady state at date 0, and that agents learn that productivity A will permanently increase from date T on. Such an announcement corresponds to anticipated technological change, which is at the heart of a vast literature on news shocks.

We start by a comparison of steady states. According to question 7., the jump in A implies a rise in the steady-state capital stock. This is intuitive. A necessary condition for optimality is that, at all dates, the marginal cost of investing equals the marginal product of capital. *Ceteris paribus*, the announced rise in A implies that the steady-state marginal product of capital will increase, while the steady-state marginal cost of investment is constant. It follows that the steady-state capital stock must increase to restore the equality (the marginal product of capital is decreasing in K_t).



We now turn to the dynamics of the adjustment toward the new steady state. When agents learn that A will increase permanently, they understand that the discounted sum of future marginal products of capital has just jumped. From equation (14), q_t rises instantaneously. Since investment is an increasing function of q, I_t also jumps at date 0. Moreover, we know that from date T on, the economy must be on its new saddle path (otherwise, it would not converge to its new steady state), and that there cannot be any anticipated jump in consumption (this would violate the Euler equation for consumption, obtained below). This means that the only period at which variables may jump is date 0, when agents reoptimize. After this initial jump, the economy is away from its initial steady state, and the capital stock increases progressively as the economy moves toward its new saddle path.

An easy way to analyze this process is to use a phase diagram. The rise in A leaves the $\{dk_t = 0\}$ locus unchanged, but modifies the $\{dq_t = 0\}$ locus by shifting it to the right. For the reasons discussed above, this shift leads to a higher new steady-state capital stock. The initial jump in q_t is represented by the red arrow, and the subsequent dynamics of the economy toward its new saddle path is represented by the blue arrow. The size of the initial jump ensures that the economy reaches this path precisely at date T.

10. As discussed before, the optimal investment strategy maximizes the set of available consumption trajectories for the representative household. Hence, letting

$$W^* = \max_{I_t} \int_0^\infty e^{-rt}[Y_t - I_t - f(I_t)]dt,$$

the household's problem writes

$$\begin{aligned} \max_{C_t} \quad & \int_0^\infty e^{-\rho t} C_t^\beta dt \\ \text{s.t.} \quad & \int_0^\infty e^{-rt} C_t dt \leq W_0 + W^*, \\ & C_t \geq 0. \end{aligned}$$

11. This is now a static problem. Neglecting the non-negativity constraint on consumption, necessary and sufficient optimality conditions are

$$\begin{aligned} e^{-\rho t}\beta C_t^{\beta-1} &= \mu e^{-rt}, \\ \int_0^\infty e^{-rt} C_t dt &= W_0 + W^*, \\ C_t &\geq 0. \end{aligned}$$

Solving this system yields

$$\begin{aligned} C_t &= \left(\frac{\beta}{\mu}\right)^{\frac{1}{1-\beta}} = \overline{C}, \\ \overline{C} &= r(W_0 + W^*), \end{aligned}$$

so the optimal behavior is to keep consumption constant. This is not be surprising: in the Ramsey model, the Euler equation equates the growth rate of consumption to the difference between the real interest rate and the discount factor. In this economy, this difference is equal to zero, since $r = \rho$.

12. We assume that at date 0, the economy is at its initial steady state with $W_0 = 0$. Hence, the trade balance writes

$$TB_0 = AK_0^\alpha - C_0 - I_0 - f(I_0) = A\left(\frac{\alpha A}{r}\right)^{\frac{\alpha}{1-\alpha}} - rW^*,$$

since $I_0 = 0$ and $C_0 = \overline{C}$. When the future rise in A is announced, both I_0 and W^* increase, so the trade balance deteriorates instantaneously.

3 A Quick Review on Bellman Equation: Simple RBC Model

This answer is from previous year's TA Alban Moura.

1. Define the value function $V(K_t, A_t)$ as the date-t expected present value of lifetime utility along the optimal trajectory, given initial conditions (K_t, A_t):

$$V(K_t, A_t) := \max_{(C_{t+s}, L_{t+s}, K_{t+1+s})_{s\geq 0}} \left\{ E_t \sum_{s=0}^{\infty} \rho^s [\log C_{t+s} + b \log(1 - L_{t+s})] \right\}$$

subject to

$$\begin{aligned}
Y_{t+s} &= K_{t+s}^{\alpha} (A_{t+s} L_{t+s})^{1-\alpha}, \\
K_{t+s+1} &= Y_{t+s} - C_{t+s}, \\
\log A_{t+s+1} &= \rho_A \log A_{t+s} + \epsilon_{t+s+1}, \\
A_t, K_t &\ \text{ given.}
\end{aligned}$$

Keeping the constraints implicit, we can manipulate V to obtain

$$\begin{aligned}
V(K_t, A_t) &= \max_{(C_{t+s}, L_{t+s}, K_{t+1+s})_{s\geq 0}} \left\{ E_t \sum_{s=0}^{\infty} \rho^s [\log C_{t+s} + b \log(1 - L_{t+s})] \right\} \\
&= \max_{(C_{t+s}, L_{t+s}, K_{t+1+s})_{s\geq 0}} \left\{ \log C_t + b \log(1 - L_t) \right. \\
&\qquad\qquad \left. + E_t \sum_{s=1}^{\infty} \rho^s [\log C_{t+s} + b \log(1 - L_{t+s})] \right\} \\
&= \max_{(C_{t+s}, L_{t+s}, K_{t+1+s})_{s\geq 0}} \left\{ \log C_t + b \log(1 - L_t) \right. \\
&\qquad\qquad \left. + \rho E_t \sum_{s=0}^{\infty} \rho^s [\log C_{t+1+s} + b \log(1 - L_{t+1+s})] \right\} \\
&= \max_{C_t, L_t, K_{t+1}} \left\{ \log C_t + b \log(1 - L_t) + \rho E_t V(K_{t+1}, A_{t+1}) \right\}.
\end{aligned}$$

The last equality is called a Bellman equation. It reduces the original infinite-horizon optimization problem into a two-period problem consisting of choosing between a contemporaneous payoff and a continuation value, and the equivalence between these two problems is called the 'principle of optimality'. There are of course mathematical technicalities that one needs to worry about when replacing the original optimization problem by the Bellman equation, but in simple RBC models as this one, things just work fine.

2. Given the log-linear structure of the economy, we guess that the value function writes

$$V(K, A) = \beta_0 + \beta_K \log K + \beta_A \log A,$$

where the β's are unknown real constants. This guess implies that

$$E_t V(K_{t+1}, A_{t+1}) = \beta_0 + \beta_K \log(Y_t - C_t) + \rho_A \beta_A \log A_t,$$

using the capital accumulation equation and the law of motion of A_t. We can therefore write the Bellman equation as

$$V(K_t, A_t) = \max_{C_t, L_t} \left\{ \log C_t + b \log(1 - L_t) + \rho[\beta_0 + \beta_K \log(K_t^\alpha (A_t L_t)^{1-\alpha} - C_t) + \rho_A \beta_A \log A_t] \right\}.$$

3. The first-order condition for consumption is

$$\frac{1}{C_t} = \frac{\rho\beta_K}{Y_t - C_t} \qquad \Longleftrightarrow \qquad \frac{C_t}{Y_t} = \frac{1}{1 + \rho\beta_K},$$

implying a constant consumption share in output.

4. The first-order condition for labor supply is

$$\frac{b}{1 - L_t} = \rho\beta_K \frac{(1-\alpha) Y_t / L_t}{K_{t+1}} \qquad \Longleftrightarrow \qquad \frac{bL_t}{1 - L_t} = \rho\beta_K (1-\alpha) \frac{Y_t}{I_t},$$

using that $K_{t+1} = I_t$. Now, using the expression for C_t / Y_t and that $I_t + C_t = Y_t$, it is straightforward to see that

$$\frac{I_t}{Y_t} = \frac{\rho\beta_K}{1 + \rho\beta_K}.$$

It follows that

$$\frac{bL_t}{1 - L_t} = \frac{\rho\beta_K (1-\alpha)(1 + \rho\beta_K)}{\rho\beta_K} \qquad \Longleftrightarrow \qquad L_t = \frac{(1-\alpha)(1+\rho\beta_K)}{b + (1-\alpha)(1+\rho\beta_K)} = L^*,$$

so that labor supply is constant along the optimal trajectory. This is due to logarithmic preferences, which exactly balance wealth and substitution effects on labor supply.

5. We are now in position to verify the guess for the value function. Letting a star denote optimal choices, we must have

$$V(K_t, A_t) = \log C_t^* + b \log(1 - L^*) + \rho[\beta_0 + \beta_K \log(Y_t^* - C_t^*) + \rho_A \beta_A \log A_t].$$

From the equilibrium relation between C and Y, we also have

$$\begin{aligned} \log C_t^* &= \xi_1 + \alpha \log K_t + (1-\alpha) \log A_t, \\ \log(Y_t^* - C_t^*) &= \xi_2 + \alpha \log K_t + (1-\alpha) \log A_t, \end{aligned}$$

where $\xi_1 = -\log(1 + \rho\beta_K) + (1-\alpha) \log L^*$ and $\xi_2 = \log(\rho\beta_K) + \xi_1$ are two constants. Hence, we can write the value function as

$$V(K_t, A_t) = \beta_0' + \underbrace{\alpha(1 + \rho\beta_K)}_{\beta_K'} \log K_t + \underbrace{[(1-\alpha)(1+\rho\beta_K) + \rho\rho_A\beta_A]}_{\beta_A'} \log A_t,$$

where $\beta_0' = \xi_1 + b\log(1+L^*) + \rho(\beta_0 + \beta_K\xi_2)$.

6. To verify the guess, we just have to find reals β_0, β_K, and β_A such that $\beta_0 = \beta_0'$, $\beta_K = \beta_K'$, and $\beta_A = \beta_A'$. Neglecting the constant β_0, which is cumbersome to compute and meaningless from an economic point of view, simple algebra yields

$$\beta_K = \frac{\alpha}{1-\alpha\rho}, \qquad \beta_A = \frac{1-\alpha}{(1-\alpha\rho)(1-\rho_A\rho)}.$$

The existence of a solution validates our guess for the value function.

4 A Quick Review on Bellman Equation: Growth

1– The only thing that restricts the consumer's choice in period t is his current wealth, because of the no borrowing constraint and the fact that she is not receiving income at t. Therefore, the only state variable is her current wealth A_t, which is endogenous since it depends on the past choices of consumption. There are no exogenous state variables in this formulation.

2– To get the payoff function, we need to express time-t instantaneous utility as a function of the time-t state variables and the time-(t+ 1) endogenous state variables, that are chosen at t. Now, observe that $C_t = A_t - A_{t+1}/R$. Therefore,

$$F(A_t, A_{t+1}) = u(c_t) = u(A_t - A_{t+1}/R)$$

2– The no borrowing constraint implies that the consumer cannot choose amounts of consumption that are greater than her current wealth. This is the same as saying that she cannot become a net debtor, although she can consume above her current income (which is zero or maybe interest payments) in the sense that she can consume her assets. On the other hand, she cannot consume a negative amount, and therefore her wealth next period cannot exceed the amount that corresponds to saving everything and consuming zero today. That is to say,

$$0 \leq A_{t+1} \leq RA_t$$

4– With all these elements, we can write down the consumer's Bellman equation as

$$v(A) = \max_{0 \leq A' \leq RA} u(A - A'/R) + \beta v(A')$$

5– Now the state variables are Y and A. The new borrowing constraint is

$$0 \leq A_{t+1} \leq RA_t + Y_t$$

Therefore,

$$v(A, Y) = \max_{0 \leq A' \leq RA+Y} u(A + Y/R - A'/R) + \beta v(A', \mu Y)$$

6– We can write the Bellman equation as

$$v(\alpha A, \alpha Y) = \max_{0 \leq A' \leq R\alpha A+\alpha Y} u(\alpha A + \alpha Y/R - A'/R) + \beta v(A', \mu\alpha Y)$$

Plugging in the CRRA utility function, we have,

$$v(\alpha A, \alpha Y) = \max_{0 \le A' \le R\alpha A + \alpha Y} \frac{1}{1-\theta} (\alpha A + \alpha Y/R - A'/R)^{1-\theta} + \beta v(A', \mu\alpha Y)$$

$$v(\alpha A, \alpha Y) = \max_{0 \le \frac{A'}{\alpha} \le RA + Y} \frac{1}{1-\theta} (\alpha A + \alpha Y/R - \alpha(\frac{A'}{\alpha R}))^{1-\theta} + \beta v(\alpha(\frac{A'}{\alpha}), \mu\alpha Y)$$

Denote $\tilde{A}' = \frac{A'}{\alpha}$ we have

$$v(\alpha A, \alpha Y) = \max_{0 \le \tilde{A}' \le RA + Y} \frac{\alpha^{1-\theta}}{1-\theta} (A + Y/R - \tilde{A}'/R)^{1-\theta} + \beta v(\alpha \tilde{A}', \mu\alpha Y)$$

This means the value function is homogeneous of degree $1-\theta$ in A and Y. i.e.

$$v(\alpha A, \alpha Y) = \alpha^{1-\theta} v(A, Y)$$

Let $\alpha = \frac{1}{Y}$. We have,

$$v(A/Y, 1) = \max_{0 \le A' \le RA/Y + 1} \frac{(1/Y)^{1-\theta}}{1-\theta} (A + Y/R - A'Y/R)^{1-\theta} + \beta v(A', \mu)$$

$$v(A/Y, 1) = \max_{0 \le A' \le RA/Y + 1} \frac{1}{1-\theta} (A/Y + 1/R - A'/R)^{1-\theta} + \beta v(A', \mu)$$

Let $\alpha = \mu$.

$$v(A', \mu) = \mu^{1-\theta} v(A'/\mu, 1) \equiv \mu^{1-\theta} v(A'', 1)$$

with $A' = \mu A''$

So the Bellman equation becomes

$$v(A/Y, 1) = \max_{0 \le A'' \le (R/\mu)A/Y + 1/\mu} \frac{1}{1-\theta} (A/Y + 1/R - \mu A''/R)^{1-\theta} + \beta \mu^{1-\theta} v(A'', 1)$$

Denote $v(A/Y, 1) \equiv V(a)$, where $a = A/Y$,

$$V(a) = \max_{0 \le A'' \le (R/\mu)a + 1/\mu} \frac{1}{1-\theta} (a + 1/R - \mu A''/R)^{1-\theta} + \beta \mu^{1-\theta} V(A'')$$

So A/Y is the only state variable.

7– The condition for discounting is $\beta \mu^{1-\theta} < 1$. It is reasonable. It is always satisfied as long as $\beta < 1$, $\mu > 1$ and $\theta \ge 1$.

5 Continuous Time Bellman Equation

1– We have the Discrete Time Bellman Equation from t to $t + \Delta t$ as

$$V(x, t) = max_u \{w(x, u, t)\Delta t + e^{-\rho \Delta t} EV(x', t')\}$$

2– The Bellman Equation is equivalent to

$$V(x,t) = max_u\{w(x,u,t)\Delta t + (1+\rho\Delta t)^{-1}EV(x',t')\}$$

$$(1+\rho\Delta t)V(x,t) = max_u\{(1+\rho\Delta t)w(x,u,t)\Delta t + EV(x',t')\}$$

$$\rho\Delta t V(x,t) = max_u\{(1+\rho\Delta t)w(x,u,t)\Delta t + EV(x',t') - V(x,t)\}$$

As $\Delta t \to 0$, we could wipe out terms in Δt^2.

$$\rho\Delta t V(x,t) = max_u\{w(x,u,t)\Delta t + E_t[V(x',t') - V(x,t)]\}$$

Divided by Δt and take limits, we have

$$\rho V(x,t) = max_u\{w(x,u,t) + \frac{dV}{dt}\}$$

(I am not clear about this point in class. The continuous time Bellman equation contains the derivative in respect to t, not the partial derivative. We could think the value function as a function only on t, instead of thinking it as a function on $x(t)$ and t. $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt}$)

3– The Bellman Equation in Q2 is equivalent to

$$\rho\Delta t V(x,t) = max_u\{w(x,u,t)\Delta t + E(dV)\}$$

$Edz = 0$. So $E(\frac{\partial V}{\partial x}bdz) = 0$. Using Ito's lemma, we get

$$\rho\Delta t V(x,t) = max_u\{w(x,u,t)\Delta t + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b^2]\Delta t\}$$

$$\rho V(x,t) = max_u\{w(x,u,t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b^2]\}$$

We have a second order ordinary differential equation (ODE). In general a large class of functions are consistent with this ODE. To pin down a solution we need to know something about the economics of the value function. This will provide constraints that pick out a single solution.

Value Matching: All continuous problems must have continuous value functions. So the value of the value function and the boundary are the same at the stopping point.

Smooth Pasting: stopping at our optimal value of the state variable is optimal. i.e. it gives higher utility than stopping earlier or later. So there canâĂŹt be a kink at the boundary. So the derivatives (w.r.t. x) are the same for the value function and the boundary function at the stopping point.

An introduction to stochastic dynamic optimization

This appendix is from previous year's TA Alban Moura. Though we discuss deterministic case in the growth part. We will move to stochastic case later. You could start reading it now or wait until we discuss stochastic case.

This appendix provides an introduction to stochastic dynamic optimization in discrete time. For a thorough discussion of the mathematical apparatus I keep hidden in the derivations, you should browse the well-known book by Stokey, Lucas, and Prescott (1989). Alternatively, Acemoglu (2009, chapter 16) reviews the main results in an intuitive fashion.

I consider a simple canonical problem with a single control variable c_t, a single endogenous state variable s_t, and a single stochastic process z_t. The extension to more complex problems is straightforward. Specifically, the problem writes

$$\max_{(c_{t+j}, s_{t+j+1})_{j\geq 0}} V(s_t, z_t) = E_t \sum_{j=0}^{\infty} \beta^j U(c_{t+j}, z_{t+j}) \tag{$\mathcal{P}_t$}$$

subject to

$$s_{t+j+1} = G(c_{t+j}, s_{t+j}, z_{t+j}),$$

to the law of motion of z_t, and to an initial condition (s_t, z_t). I assume that the flow payoff function $U(c, z)$ is increasing in c and that the function $G(c, s, z)$ representing the law of motion of the endogenous state variable is decreasing in c and increasing in s. E_t denotes the expectation operator conditional to all relevant information available at date t.

Below, I describe two equivalent approaches to characterize the solution to the maximization problem ($\mathcal{P}_t$). A first possibility is to use Lagrange multipliers, which has the advantage of simplicity. However, it requires some care in dealing properly with expectations. The second approach exploits the recursive structure of ($\mathcal{P}_t$) through dynamic programming.

Lagrange multipliers

Letting λ_{t+j} denote the multiplier on the law of motion of the state variable at date $t+j$, we can associate ($\mathcal{P}_t$) with a Lagrangian function given by

$$\mathcal{L}_t = E_t \sum_{j=0}^{\infty} \beta^j \Big[U(c_{t+j}, z_{t+j}) + \lambda_{t+j} [G(c_{t+j}, s_{t+j}, z_{t+j}) - s_{t+j+1}] \Big].$$

The FOCs with respect to c_{t+j} and s_{t+j+1} write

$$E_t \frac{\partial U(c_{t+j}, z_{t+j})}{\partial c_{t+j}} = -E_t \lambda_{t+j} \frac{\partial G(c_{t+j}, s_{t+j}, z_{t+j})}{\partial c_{t+j}}, \tag{15}$$

$$E_t \lambda_{t+j} = \beta E_t \lambda_{t+j+1} \frac{\partial G(c_{t+j+1}, s_{t+j+1}, z_{t+j+1})}{\partial s_{t+j+1}}. \tag{16}$$

It is now key to recognize that these FOCs have different implications for $j = 0$ or $j \geq 1$. When solving ($\mathcal{P}_t$) at date t, the only variables that are effectively chosen are c_t and s_{t+1}. Future variables such as c_{t+1}, s_{t+2}, c_{t+2}, s_{t+3}, ..., indeed enter the system (1)-(2), but only in terms of *forecasts*. In effect, c_{t+1} and s_{t+2} will be chosen at date $t+1$ given all information available at $t+1$ by solving ($\mathcal{P}_{t+1}$), c_{t+2} and s_{t+3} will be chosen at date $t+2$ given all information available at $t+2$ by solving ($\mathcal{P}_{t+2}$), and so on. Since the information set changes over time, the FOCs at date t are not useful to pin down these future variables. Overall, the only FOCs we need to care about here are those for $j = 0$, which write

$$\frac{\partial U(c_t, z_t)}{\partial c_t} = -\lambda_t \frac{\partial G(c_t, s_t, z_t)}{\partial c_t}, \qquad \lambda_t = \beta E_t \lambda_{t+1} \frac{\partial G(c_{t+1}, s_{t+1}, z_{t+1})}{\partial s_{t+1}}.$$

We can consolidate them by elimination of the Lagrange multiplier to obtain the Euler equation

$$\frac{\partial U(c_t, z_t)/\partial c_t}{\partial G(c_t, s_t, z_t)/\partial c_t} = \beta E_t \frac{\partial U(c_{t+1}, z_{t+1})/\partial c_{t+1}}{\partial G(c_{t+1}, s_{t+1}, z_{t+1})/\partial c_{t+1}} \frac{\partial G(c_{t+1}, s_{t+1}, z_{t+1})}{\partial s_{t+1}}, \tag{17}$$

which imposes an equality between a contemporaneous payoff on the left-hand side and a discounted expected future payoff on the right-hand side. The solution to ($\mathcal{P}_t$) is then characterized by the Euler equation (3), by the law of motion of the endogenous state variable, by the law of motion of the exogenous stochastic process z, and by a transversality condition that I omit here.

Dynamic programming

The dynamic programming approach replaces ($\mathcal{P}_t$) by the following functional equation:

$$V(s_t, z_t) = \max_{c_t, s_{t+1}} \left[U(c_t, z_t) + \beta E_t V(s_{t+1}, z_{t+1}) \right] \tag{18}$$

subject to

$$s_{t+1} = G(c_t, s_t, z_t)$$

and the law of motion of z_t. Clearly, some regularity conditions are needed for the solution to equation (4) to be equivalent for the solution to ($\mathcal{P}_t$). However, I do not discuss this here.

Although we don't know the form of the value function $V(s, z)$, we can still perform the maximization with respect to c_t and s_{t+1}. Eliminating s_{t+1} by plugging the constraint into (4), the FOC writes

$$\frac{\partial U(c_t, z_t)}{\partial c_t} = -\beta \frac{\partial G(c_t, s_t, z_t)}{\partial c_t} E_t \frac{\partial V(s_{t+1}, z_{t+1})}{\partial s_{t+1}}. \tag{19}$$

Also, differentiating (4) with respect to s_t holding c_t constant gives

$$\frac{\partial V(s_t, z_t)}{\partial s_t} = \beta \frac{\partial G(c_t, s_t, z_t)}{\partial s_t} E_t \frac{\partial V(s_{t+1}, z_{t+1})}{\partial s_{t+1}}. \tag{20}$$

Comparing (5) and (6) shows that

$$\frac{\partial U(c_t, z_t)/\partial c_t}{\partial G(c_t, s_t, z_t)/\partial c_t} = -\beta E_t E_t \frac{\partial V(s_{t+1}, z_{t+1})}{\partial s_{t+1}} = -\frac{\partial V(s_t, z_t)/\partial s_t}{\partial G(c_t, s_t, z_t)/\partial s_t}.$$

The same equation at date $t + 1$ gives

$$\frac{\partial U(c_{t+1}, z_{t+1})/\partial c_{t+1}}{\partial G(c_{t+1}, s_{t+1}, z_{t+1})/\partial c_{t+1}} = -\frac{\partial V(s_{t+1}, z_{t+1})/\partial s_{t+1}}{\partial G(c_{t+1}, s_{t+1}, z_{t+1})/\partial s_{t+1}}. \tag{21}$$

We can use this result to eliminate $\partial V(s_{t+1}, z_{t+1})/\partial s_{t+1}$ from (5), to get

$$\frac{\partial U(c_t, z_t)/\partial c_t}{\partial G(c_t, s_t, z_t)/\partial c_t} = \beta E_t \frac{\partial U(c_{t+1}, z_{t+1})/\partial c_{t+1}}{\partial G(c_{t+1}, s_{t+1}, z_{t+1})/\partial c_{t+1}} \frac{\partial G(c_{t+1}, s_{t+1}, z_{t+1})}{\partial s_{t+1}}. \tag{22}$$

This is exactly the same equation as (3), so that an approach based on Lagrange multipliers delivers the same results as dynamic programming.