TSE Master 2 — Macroeconomics I
Problem Set 2

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1 Cake-Eating Problem

1. The social planner's problem is:

$$
\begin{aligned}
\max_{\{C_t,K_t\}_{t=0}^{+\infty}} \quad & \beta^t log(C_t) & (1)\\
\text{s.t.} \quad K_{t+1} &= Y_t - C_t, & (2)\\
Y_t &= F(K_t) = K_t & (3)\\
K_t &\geq 0, \ \ K_0 \ \text{given} & (4)
\end{aligned}
$$

where K_0 given is the initial endowment of this economy.

2. The Bellman equation is

$$V(K_t) = \max_{C_t}\{log(C_t) + \beta V_{K_{t+1}}\}$$

Then we derive the F.O.C and the envelope condition:

$$0 = \frac{1}{C_t} + \beta V'(K_{t+1}) \tag{5}$$

$$V(K_t) = \beta V'(K_{t+1}) \cdot \frac{K_{t+1}}{K_t} \tag{6}$$

From (2) and (3), we get:

$$K_{t+1} = K_t - C_t$$

Therefore, we have $\frac{\partial K_{t+1}}{\partial K_t} = 1$, so (6) equals to:

$$V'(K_t) = \beta V'(K_{t+1})$$

*Thanks to Alban for providing solutions from previous years. Please let us know if you find typos or errors: lan.lansl@yahoo.fr

From equation (5), we have that:

$$C_{t+1} = \beta C_t$$

equally,

$$U'(C_t) = \beta U'(C_{t+1})$$

This is the Euler equation.

In standard Ramsey model, we have:

$$U'(C_t) = \beta U'(C_{t+1})(F'(K_t) + 1 - \delta)$$

Here in this problem, we have $F'(K_t) = \dfrac{\partial Y_t}{K_t} = 1$, and $1 - \delta = 0$ according to full depreciation. $F'(K_t) + 1 - \delta$ is called the marginal product of capital net of depreciation.

3. Saving in period t equals to:

$$S_t = Y_t - C_t = K_t - C_t = Y_{t+1}$$

So saving in period t equals to production in the following period. This is the intuition for "cake-eating" economy. This can be easily understood by looking at the graph below:

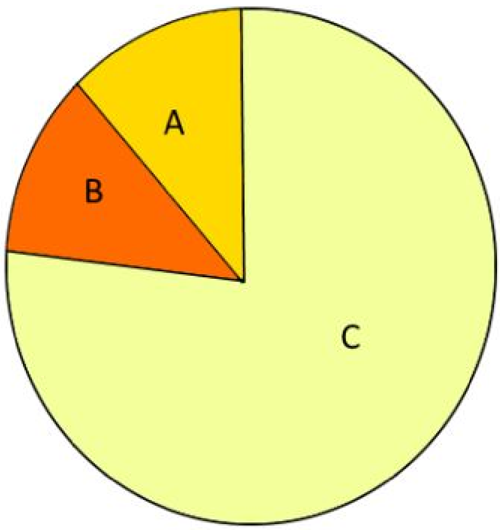

Figure 1: Cake-Eating Economy

The economy starts from period $t = 0$ with the initial endowment $K_0 = A + B + C$, and consumes $C_0 = A$ in that period. So the economy is left with $K_1 = K_0 - C_0$, which equals the sum of B and C. Then period 0 ends, and we move to period 1. In period 1, the economy has total production of K_1, and consumes $C_1 = B$, so the economy is left with C in period 2, and so on.

4. According to Euler equation, we know that consumption keeps decreasing with a rate of β. So

we do not have steady state, or do we have Modified Golden Rule. The fact that the economy keeps contracting can be seen in the evolvment of Y_t.

$$Y_t = K_t = K_{t-1} - C_{t-1}$$

Since $C_t > 0$, so $Y_t < Y_{t-1}$. Therefore the economy keeps shrinking.

Habit Formation in Growth Model

1. The utility takes the form of:

$$U(x) = \frac{x^{1-\sigma}}{1-\sigma}$$

Therefore the relative risk aversion is:

$$-\frac{xU''(x)}{U'(x)} = -\frac{x \cdot [-\sigma x^{-\sigma-1}]}{x^{-\sigma}} = \sigma > 1$$

σ is the risk aversion of the agent. It captures the concavity of the utility function. The more concave it is, the more risk averse the agent is.

2. The law of motion (LOM) of habit is:

$$\dot{H}_t = \eta(C_t - H - t)$$

It can be rewritten as the following in discrete timeïijŻ

$$H_{t+1} - H_t = \eta(C - t - H_t)$$

which equals that

$$H_{t+1} = \eta C_t + (1-\eta)H_t$$

So η is the weight of consumption in dynamics of capital stock.

If $\eta = 0$, there is full weight on historical stock, which means consumers do not update habit stock in each period according to their consumption.

If $\eta = 1$, there is full weight on consumption, which means consumers fully update habit stock in each period.

Then we turn to the utility function to understand γ. γ is the importance of habit stock relative to consumption in forming utility.

If $\gamma = 0$, the habit stock does not have an influence on utility.

If $\gamma = 1$, only the ratio of consumption over habit is important for the consumers. Changing the level of consumption or habit while anchoring their ratio does not make any difference.

To understand the economic intuition of habit H_t, we study the marginal utility of habit:

$$MU_{H,t} = -\gamma \frac{C_t^{1-\sigma}}{H_t^{\gamma(1-\sigma)+1}} < 0$$

The intuition for this negative partial effect is: given consumption level across households, when the habit stock is high, then the households are quite "used to" high level of consumption through the life path. Therefore, they do not gain as much utilities from consuming as the households with lower habit stock. consumption:

$$MU_{C,t} = \frac{\partial U(C_t, H_t)}{\partial C_t} = \frac{C_t^{-\sigma}}{H_t^{\gamma(1-\sigma)}} > 0$$

Then we differentiate marginal utility of consumption on habit:

$$\frac{\partial MU_{C,t}}{\partial H_t} = \gamma(\sigma - 1) C_t^{-\sigma} H_t^{-\gamma(1-\sigma)} > 0$$

This means: given same consumption level across households, the households who have higher habit level will tend to consume more than the households who have lower habit level, simply because higher habit leads to higher marginal utility of consumption.

3. The choice variables is consumption C_t, and the state variables are capital stock K_t and habit stock H_t. The social planner problem is:

$$\max_{\{C_t, H_t, K_t\}_{t=0}^{+\infty}} \quad \int_t e^{\rho t} U(C_t) dt \tag{7}$$

$$\text{s.t.} \quad \dot{H}_t = \eta(C_t - H_t) \tag{8}$$

$$\dot{K}_t = I_t - \delta K_t \tag{9}$$

$$Y_t = A_t K_t \tag{10}$$

$$K_t \geq 0, \;\; K_0 \;\text{ given.} \tag{11}$$

4. According to the definition of "outward-looking", the LOM of habit stock can be rewritten as:

$$\dot{H}_t = \eta(\mathbb{C}_t - H_t)$$

4-a. The aggregate level of consumption is determined this way:

$$\mathbb{C}_t = \int_i C_{ti} di = C_{ti} = C_t$$

Here, $C_{ti} = C_i$ is due to the representative agent assumption. The effect of individual consumption on aggregate consumption is:

$$\frac{\partial \mathbb{C}_t}{\partial C_t} = 0$$

4-b. From the above, we know that aggregate consumption level equals individual consumption level.

5. The Bellman equation writes as:

$$\rho V(K_t, H_t) = \max_{C_t}\{U(C_t, H_t) + \frac{dV(K_t, H_t)}{dt}\}$$

The optimal conditions are:

$$\frac{C_t^{-\sigma}}{H_t^{\gamma(1-\sigma)}} = V_K + V_H \frac{\partial \dot{H}_t}{\partial C_t} \tag{12}$$

$$\rho V_K = \dot{V}_K + V_K(A_t - \delta) \tag{13}$$

$$\rho V_H = U_H + \dot{V}_H - \eta V_H \tag{14}$$

Rearranging terms yields that:

$$\frac{C_t^{-\sigma}}{H_t^{\gamma(1-\sigma)}} = V_K \tag{15}$$

$$\frac{\dot{V}_K}{V_K} = \rho + \delta - A_t \tag{16}$$

$$\frac{\dot{H}_t}{H_t} = \eta(\frac{C_t}{H_t} - 1) \tag{17}$$

Equation (15) implies that:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}(A_t - \delta - \rho + \gamma(\sigma - 1)\frac{\dot{H}_t}{H_t}) \tag{18}$$

Plug equation (17) in (18), we have:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\sigma}[A_t - \delta - \rho + \gamma(\sigma - 1)\eta(\frac{C_t}{H_t} - 1)] \tag{19}$$

This is the Euler Equation.

6. Capital does not matter when we study the dynamics of consumption, because capital does not play a role in the Euler Equation. This result comes from the fact that production function is linear in capital, therefore marginal product of capital does not depends on capital.

7. From the Equation (18), we see that $\frac{\dot{C}_t}{C_t}$ and $\frac{\dot{H}_t}{H_t}$ can not be zero at the same time. So we do not have the steady state as in Ramsey Model.

8. If $[\dot{\frac{C_t}{H_t}}] = 0$, then we have

$$\begin{aligned}\frac{\dot{C}_t H_t - C_t \dot{H}_t}{H_t^2} &= \frac{\dot{C}_t}{H_t} - \frac{\dot{H}_t C_t}{H_t^2} \\ &= \frac{C_t}{H_t}(\frac{\dot{C}_t}{C_t} - \frac{\dot{H}_t}{H_t}) \\ &= \frac{C_t}{H_t}[\frac{\dot{C}_t}{C_t} - \eta(\frac{C_t}{H_t} - 1)] = 0\end{aligned}$$

So we have:

$$\frac{\dot{C}_t}{C_t} = \eta(\frac{C_t}{H_t} - 1) \tag{20}$$

Together with equation (19), we have:

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{H}_t}{H_t} = \frac{A_t - \rho - \delta}{\gamma(1-\sigma) + \sigma} \tag{21}$$

$$\frac{C_t}{H_t} = 1 + \frac{1}{\eta}[\frac{A_t - \rho - \delta}{\gamma(1-\sigma) + \sigma}] \tag{22}$$

These are also the conditions which characterize Balanced Growth Path.

Dencentralization

In this problem, I provide the Hamilton method.

We have so far solved planner's problems. In this exercise, we study the decentralized competitive equilibrium of the Ramsey economy and show that it coincides with the planner's solution.

1. At each date, consumers earn $r_t W_t$ from asset holdings, $w_t L_t$ from labor income, and $D_t = Y_t - I_t - w_t L_t$ as dividends from the firm. Therefore, disposable income is

$$DI_t = r_t W_t + Y_t - I_t.$$

In practice, the only available asset in this economy is a riskless bond, in zero net supply. Therefore, $W_t = 0$ in equilibrium, so that

$$DI_t = Y_t - I_t = C_t,$$

and the representative consumer's disposable income is equal to C_t. It is not equal to Y_t because in this model, investment expenditures are supported by firms and not by households.

2. We will use standard arbitrage arguments to establish the law of motion of the value of a firm. Because of constant returns to scale in the production function, the size of firms is indeterminate and it is sufficient to focus on a representative firm. The general form of an asset pricing equation

is

$$\text{Rate of return} \times \text{Value} = \text{Dividend} + \text{Expected capital gains},$$

where the left-hand side represents the payoff from selling the asset and lending the cashflow on the market and the right-hand side represents the payoff from holding the asset, receiving the dividend, and being exposed to potential changes in the asset price. This (Bellman) equation just states that at any optimum, investors should be indifferent between the two alternatives.

Consider now the firm as an asset between dates t and $t + dt$. We have

$$\begin{aligned} \text{Rate of return} &= r_t dt, \\ \text{Value of the firm at } t &= V_t, \\ \text{Dividend} &= (Y_t - I_t - w_t L_t)dt, \\ \text{Capital gains} &= V_{t+dt} - V_t. \end{aligned}$$

The asset pricing equation is thus

$$r_t dt V_t = (Y_t - I_t - w_t L_t)dt + V_{t+dt} - V_t \quad \iff \quad r_t V_t = \frac{V_{t+dt} - V_t}{dt} + Y_t - I_t - w_t L_t.$$

Taking $dt \to 0$, we get

$$r_t V_t = \frac{dV_t}{dt} + Y_t - I_t - w_t L_t, \tag{23}$$

as was to be shown.

3. Equation (23) is an ordinary differential equation in V_t. To obtain the solution, multiply (23) by the integrating factor $\exp(-\int_0^t r_\tau d\tau)$ to get

$$\begin{aligned} &\exp\left(-\int_0^t r_\tau d\tau\right)\frac{dV_t}{dt} - \exp\left(-\int_0^t r_\tau d\tau\right) r_t V_t = -\exp\left(-\int_0^t r_\tau d\tau\right)(Y_t - I_t - w_t L_t) \\ \iff \quad &\frac{d}{dt}\left[\exp\left(-\int_0^t r_\tau d\tau\right) V_t\right] = -\exp\left(-\int_0^t r_\tau d\tau\right)(Y_t - I_t - w_t L_t). \end{aligned}$$

Integrating between dates t and T yields

$$\exp\left(-\int_0^T r_\tau d\tau\right) V_T - \exp\left(-\int_0^t r_\tau d\tau\right) V_t = -\int_t^T \exp\left(-\int_0^\tau r_s ds\right)(Y_\tau - I_\tau - w_\tau L_\tau) d\tau.$$

Assume that $\exp(-\int_0^T r_\tau d\tau) V_T$ converges as $T \to \infty$ and denote B_∞ this limit. The value of the firm at date t can then be written as the sum of two terms:

$$V_t = \exp\left(\int_0^t r_\tau d\tau\right) B_\infty + \int_t^\infty \exp\left(-\int_t^\tau r_s ds\right)(Y_\tau - I_\tau - w_\tau L_\tau) d\tau.$$

The first term explodes at rate r_t: it is a bubble. The second term is the discounted value at date t of all future flows of dividend: it is the fundamental value.

4. From now on, we assume that V_t is always equal to the fundamental value of the firm, so

$B_\infty = 0$. Since the firm chooses its investment strategy to maximize V_t, its problem writes

$$\max_{I_t, K_t, L_t} \quad \int_t^\infty \exp\left(-\int_t^\tau r_s d_s\right)[F(K_\tau, L_\tau) - I_\tau - w_\tau L_\tau]d\tau$$
$$\text{s.t.} \quad \frac{dK_\tau}{d\tau} = I_\tau - \delta K_\tau,$$
$$K_\tau \geq 0,$$
$$1 \geq L_\tau \geq 0.$$

The Hamiltonian is

$$H(K_\tau, L_\tau, I_\tau, \lambda_\tau, \tau) = \exp\left(-\int_t^\tau r_s d_s\right)[F(K_\tau, L_\tau) - I_\tau - w_\tau L_\tau] + \lambda_\tau(I_\tau - \delta K_\tau).$$

Necessary and sufficient optimality conditions with respect to L_t, I_t, and K_t write

$$\frac{\partial F(K_\tau, L_\tau)}{\partial L} = w_\tau, \tag{24}$$
$$\exp\left(-\int_t^\tau r_s ds\right) = \lambda_\tau, \tag{25}$$
$$-\frac{d\lambda_\tau}{d\tau} = \exp\left(-\int_t^\tau r_s ds\right)\frac{\partial F(K_\tau, L_\tau)}{\partial K} - \delta\lambda_\tau, \tag{26}$$
$$\lim_{\tau\to\infty} \lambda_\tau K_\tau = 0. \tag{27}$$

Equation (25) implies that

$$\frac{d\lambda_\tau}{d\tau} = -r_\tau \lambda_\tau,$$

which together with (26) yields

$$\frac{\partial F(K_\tau, L_\tau)}{\partial K} = r_\tau + \delta.$$

Thus, the optimal investment strategy equates, at each period, the marginal product of capital with its marginal opportunity cost (the sum of the rental rate and the depreciation rate).

5. From the optimality condition (24), the equilibrium wage rate is equal to the marginal product of labor at each date.

6. Let W_t denote the stock of assets (bonds) hold by the representative household. It evolves according to

$$\frac{dW_t}{dt} = r_t W_t + DI_t - C_t, \tag{28}$$

which just states that changes in the consumer's asset position are equal to the difference between current disposable income and current expenditures on consumption. Since $DI_t = C_t$ and $W_t = 0$, we have $dW_t/dt = 0$ for all t, which is consistent with the bond being in zero net supply.

7. Imposing the no-Ponzi game constraint (that is in fact implied by the equilibrium condition

$W_t = 0$)

$$\lim_{t\to\infty} \exp\left(-\int_0^t r_\tau d\tau\right) W_t \geq 0,$$

we can integrate (28) forward to get

$$\int_0^\infty \exp\left(-\int_0^t r_\tau d\tau\right) C_t dt \leq W_0 + \int_0^\infty \exp\left(-\int_0^t r_\tau d\tau\right) DI_t dt.$$

This is the representative consumer's intertemporal budget constraint, stating that the present value of lifetime consumption expenditures cannot exceed initial wealth plus the present value of lifetime disposable income.

8. The consumer's problem now writes

$$\begin{aligned} \max_{C_t} \quad & \int_0^\infty e^{-\rho t} u(C_t) dt \\ \text{s.t.} \quad & \int_0^\infty \exp\left(-\int_0^t r_\tau d\tau\right) C_t dt \leq \int_0^\infty \exp\left(-\int_0^t r_\tau d\tau\right) DI_t dt, \\ & C_t \geq 0, \end{aligned}$$

where we have used the fact that $W_0 = 0$. This is a static problem, and the first-order condition for an interior solution is

$$e^{-\rho t} u'(C_t) = \mu \exp\left(-\int_0^t r_\tau d\tau\right),$$

where μ is the Lagrange multiplier on the intertemporal budget constraint. Taking time derivatives on both sides of the equation yields

$$-\rho e^{-\rho t} u'(C_t) + e^{-\rho t} u''(C_t) \frac{dC_t}{dt} = -\mu r_t \exp\left(-\int_0^t r_\tau d\tau\right).$$

Reshuffling a bit, this is also

$$\frac{dC_t}{dt} = -\frac{u'(C_t)}{u''(C_t)} (r_t - \rho).$$

Letting $\sigma(C_t) = -u'(C_t)/[C_t u''(C_t)]$ denote the intertemporal elasticity of substitution associated with the consumer's preferences, we get in the end

$$\frac{dC_t/dt}{C_t} = \sigma(C_t)(r_t - \rho),$$

which is just the Euler equation for consumption.

9. Consolidating the Euler equation for consumption and the capital accumulation equation, the

competitive equilibrium in the decentralized economy is characterized by

$$\frac{dC_t/dt}{C_t} = \sigma(C_t)\left(\frac{\partial F(K_t, L_t)}{\partial K} - \delta - \rho\right),$$
$$\frac{dK_t}{dt} = Y_t - C_t - \delta K_t,$$

together with the initial condition K_0. The same pair of equations describes the solution to the planning problem (see for example question 7. in Problem I). Therefore, the decentralized competitive equilibrium coincides with the Ramsey optimum. This is a consequence of the Second Welfare Theorem.

10. Recall that the fundamental value of the firm at date t is given by

$$V_t = \int_t^\infty \exp\left(-\int_t^\tau r_s ds\right)[F(K_\tau, L_\tau) - I_\tau - w_\tau L_\tau].$$

Using that F has constant returns to scale, that factors are paid their marginal products, and the definition of I, we have

$$F(K_\tau, L_\tau) - I_\tau - w_\tau L_\tau = (r_\tau + \delta)K_\tau + w_\tau L_\tau - \frac{dK_\tau}{d\tau} - \delta K_\tau - w_\tau L_\tau = r_\tau K_\tau - \frac{dK_\tau}{d\tau}.$$

It follows that

$$\begin{aligned} V_t &= \int_t^\infty \exp\left(-\int_t^\tau r_s ds\right)\left(r_\tau K_\tau - \frac{dK_\tau}{d\tau}\right)d\tau \\ &= \int_t^\infty \frac{d}{d\tau}\left[-\exp\left(-\int_t^\tau r_s ds\right)K_\tau\right]d\tau \\ &= -\lim_{\tau\to\infty}\exp\left(-\int_t^\tau r_s ds\right)K_\tau + K_t. \end{aligned}$$

Now, we just have to recognize that the limit of $\exp(-\int_t^\tau r_s ds)K_\tau$ is zero, which follows directly from the transversality condition (27) and from the equilibrium value of λ_τ given by (25). We finally obtain that $V_t = K_t$ for all dates t.

Knowledge Spillover

1. We know that $H_t = bK_t$, therefore $\dot{H}_t = b\dot{K}_t$. According to the LOM of K_t, we know that

$$\dot{H}_t = bF(K_t, H_t, L_t) - bC_t - \delta H_t$$

Note that in this economy, human capital scales with physical capital, therefore two state variables H_t and K_t can be simplified as one state variable K_t. Whether human capital accumulation is a side effect can be answered by checking the resource allocation of the economy $Y_t = C_t + I_t$. It is straightforward that no extra resource is allocated to accumulate human capital, therefore, human capital accumulation is a side effect, a.k.a, externality. And in this economy, we will see that social planner optimum does not equal to market equilibrium result, because market is not complete.

2. The social planner problem writes as:

$$\max_{\{C_t,H_t,K_t\}_{t=0}^{+\infty}} \quad \int_t e^{\rho t} U(C_t) dt \tag{29}$$

$$\text{s.t.} \quad \dot{K}_t = F(K_t, H_t, L_t) - C_t - \delta K_t \tag{30}$$

$$F(K_t, H_t, L_t) = K_t^{\alpha} L_t^{1-\alpha} \tag{31}$$

$$K_t \geq 0, \quad K_0 \text{ given.} \tag{32}$$

Bellman equation is:

$$\rho V(K_t) = \max_{C_t} \{ \frac{C_t^{1-\gamma}}{1-\gamma} + V(\dot{K}_t) \}$$

We know that production function $F(K_t, H_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. When social planner make marginal decision, he takes $H_t = bK_t$ into consideration. So production function becomes $F(K_t, H_t, L_t) = b^{1-\alpha} K_t L_t^{1-\alpha}$ And the optimal conditions are:

$$0 = C_t^{-\gamma} - V_K \tag{33}$$

$$\rho V_K = \dot{V}_K + V_K (b^{1-\alpha} L_t^{1-\alpha} - \delta) \tag{34}$$

Rearranging the terms yields that:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} \frac{\dot{V}_K}{V_K} \tag{35}$$

$$\frac{\dot{V}_K}{V_K} = \rho + \delta - b^{1-\alpha} L_t^{1-\alpha} \tag{36}$$

Plug (36) into (35), we have:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} [b^{1-\alpha} L_t^{1-\alpha} - \rho - \delta] \tag{37}$$

3. The utility maximization problem of the households is exactly the same as in decentralized Ramsey model, so we have:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} (r_t - \rho)$$

4. The profit maximization problem of the firm is exactly the same as in decentralized Ramsey model also, so we have:

$$\frac{\partial F(K_t, H_t, L_t)}{\partial K_t} = r_t + \delta$$

$$\frac{\partial F_{K_t, H_t, L_t}}{\partial L_t} = w_t$$

where $\frac{\partial F(K_t, H_t, L_t)}{\partial K_t}$ is the return/marginal product of physical capital, also called R_t. Therefore we have:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma}[\alpha b^{1-\alpha} L_t^{1-alpha} - \rho - \delta] \tag{38}$$

5. The growth rate of consumption can be seen in the formula above.

6. We have the Euler equation of SP and ME as:

$$\left(\frac{\dot{C}_t}{C_t}\right)_{SP} = \frac{1}{\gamma}[b^{1-\alpha} L_t^{1-\alpha} - \rho - \delta] \tag{39}$$

$$\left(\frac{\dot{C}_t}{C_t}\right)_{ME} = \frac{1}{\gamma}[\alpha b^{1-\alpha} L_t^{1-alpha} - \rho - \delta] \tag{40}$$

Because $\alpha < 1$, then we have growth rate of consumption under market equilibrium is lower than the growth rate of consumption under social planner optimum.