Macroeconomics II – Problem Set 2

Solutions

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1 Incomplete Markets and Asset Prices

This problem investigates the effects of market incompleteness on asset pric-ing following **?**. There is a continuum of individuals with identical CRRA preferences:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

and subjective discount factor is β.

a. The standard Euler equation for each individual i and asset j will be

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \frac{u'(c^i_{t+1})}{u'(c^i_t)} \right]$$

and with our preferences specification, it becomes

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c^i_{t+1}}{c^i_t} \right)^{-\gamma} \right] \tag{1.1}$$

b. From part (a), we have

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c^i_{t+1}}{c^i_t} \right)^{-\gamma} \right] = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (\varepsilon^i_{t+1})^{-\gamma} \right] \tag{1.2}$$

Recall that we know

$$\log \varepsilon^i_{t+1} | X \sim \mathcal{N} \left(-\frac{1}{2} \sigma^2_{t+1}, \sigma^2_{t+1} \right) \tag{1.3}$$

where

$$X' = \begin{pmatrix} \sigma^2_{t+1} & \frac{c_{t+1}}{c_t} & R^j_{t+1} \end{pmatrix}$$

We can rewrite (1.2) as

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left\{ \log \left[(\varepsilon^i_{t+1})^{-\gamma} \right] \right\} \right]$$

and using the law of iterated expectations,

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \mathbb{E}_t \left[\exp \left\{ -\gamma \log \varepsilon^i_{t+1} \right\} \middle| X \right] \right]$$

From the known properties of the normal distribution we know that if $z \sim \mathcal{N}(\mu, \sigma^2)$ and c a constant, then

$$\mathbb{E}[\exp(cz)] = \exp \left\{ c\mu + \frac{1}{2} c^2 \sigma^2 \right\}$$

It then follows that

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left\{ \frac{1}{2} \gamma \sigma^2_{t+1} + \frac{1}{2} \gamma^2 \sigma^2_{t+1} \right\} \right]$$

$$= \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left\{ \frac{\gamma(1+\gamma)}{2} \sigma^2_{t+1} \right\} \right] \tag{1.4}$$

what was to be shown.

c. From part (b) we have

$$1 = \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left\{ \frac{\gamma(1+\gamma)}{2} \left[A - B \log \left(\frac{c_{t+1}}{c_t} \right) \right] \right\} \right]$$

$$= \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \exp \left\{ \frac{\gamma(1+\gamma)}{2} A \right\} \exp \left\{ -\frac{\gamma(1+\gamma)}{2} B \log \left(\frac{c_{t+1}}{c_t} \right) \right\} \right]$$

$$= \beta \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \left(\frac{c_{t+1}}{c_t} \right)^{-\frac{\gamma(1+\gamma)}{2} B} \exp \left\{ \frac{\gamma(1+\gamma)}{2} A \right\} \right]$$

$$= \beta \exp \left\{ \frac{\gamma(1+\gamma)}{2} A \right\} \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\left[\gamma + \frac{\gamma(1+\gamma)}{2} B\right]} \right] = \hat{\beta} \mathbb{E}_t \left[R^j_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\hat{\gamma}} \right]$$

which is what we wanted to prove.

d. Recall that the equity premium puzzle is a *quantitative* puzzle. From the Euler equation for a risky asset j:

$$1 = \hat{\beta} \mathbb{E}_t \left[R^j_{t+1} \right] \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\hat{\gamma}} \right] + \beta \mathbf{Cov} \left[R^j_{t+1}, \left(\frac{c_{t+1}}{c_t} \right)^{-\hat{\gamma}} \right] \tag{1.5}$$

and for a risk free rate asset, we have

$$1 = \hat{\beta} R^{rf}_{t+1} \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\hat{\gamma}} \right] \tag{1.6}$$

Note that what we need to explain is a negative covariance term in (1.5), which is consistent with the data: in booms both stock returns and consumption increase and vice versa in recessions. The issue turns out to be quantitative in the sense that to match the equity premium that emerges form the data we need a γ much higher than what the empirical studies suggest.

This comes from the fact that the covariance term is bounded by the variance of consumption. The standard model implies that what matters is the variance of aggregate consumption, but if we introduce idiosyncratic shocks as in this exercise, we can take into account the fact that consumption is much more volatile.

Note that from part (c) we can express our Euler Equation as observationally equivalent to the standard one, where the new parameters are $\hat{\gamma}$ and $\hat{\beta}$. And note that for $B > 0$ and $A > 0$ we have $\gamma < \hat{\gamma}$ and $\beta < \hat{\beta}$, which both make the premium larger. In particular, the usual **?** calculation now implies that what is required to be very high to match the data is $\hat{\gamma}$, which allows for some B to have a much lower γ, since

$$\hat{\gamma} = \gamma + \frac{1}{2}\gamma(1+\gamma)B$$

2 Idiosyncratic Investment Risk

This exercise is a simplified version of **?**. It augments the neoclassical growth model with uninsured capital-income risk. A quick precision on the exposition of the exercise : on top of investing in their *own* capital stock, households can freely trade a riskless bond, but not other financial asset.

a. Since there is a continuum of households, idiosyncratic uncertainty washes out in the aggregate. So, a competitive equilibrium can be defined as a *deterministic* sequence of wages and rates of return $\{\omega_t, R_t\}_{t=0}^{\infty}$, a *deterministic* aggregate path for consumption, capital and output $\{C_t, K_t, Y_t\}_{t=0}^{\infty}$, and a collection of the households' contingent plans $\{c_t^i, n_t^i, y_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^{\infty}$, for $i \in [0,1]$, such that the following conditions hold:

   i) *Optimality*: $\{c_t^i, n_t^i, y_t^i, k_{t+1}^i, b_{t+1}^i\}_{t=0}^{\infty}$ maximizes household i's utility for every i.

   ii) *Labor-market clearing*: for all $t \geq 0$, we have

$$\int_{i\in[0,1]} n_t^i \mathrm{d}i = 1 \tag{2.1}$$

   iii) *Bond-market clearing*: for all $t \geq 0$, we have

$$\int_{i\in[0,1]} b_t^i \mathrm{d}i = 0 \tag{2.2}$$

   iv) *Aggregation*: for all $t \geq 0$, we have

$$C_t = \int_{i\in[0,1]} c_t^i \mathrm{d}i, \qquad K_t = \int_{i\in[0,1]} k_t^i \mathrm{d}i, \qquad Y_t = \int_{i\in[0,1]} y_t^i \mathrm{d}i \tag{2.3}$$

b. If the lower bound of the support of A_t^i is zero, then since we assume that $F(0, n_t^i) = 0$, the worst possible realization of capital income is also zero. Now, if we denote the household's present value of future labor income (also referred to as "human wealth") by

$$h_t \equiv \sum_{j=1}^{\infty} \frac{\omega_{t+j}}{R_{t+1} \cdots R_{t+j}} \tag{2.4}$$

then the natural solvency constraint could be written as

$$b_{t+1}^i \geq -h_t \tag{2.5}$$

implying that even in the worst case, the household would be able to repay the debt. Notice that the right-hand side of (2.5) is the same for all $i \in [0,1]$: human wealth does not vary in the cross section. This is due to the assumption that everyone inelastically supplies one unit of labor, and that the productivity (and thus, wages) is the same across households. As we will see, this simplifies aggregation a lot.

c. Denoting by $V_t(k, b, A)$ individual's date-t value function, given capital holdings of k, bond holdings of b and current productivity realization of A, we can write the optimization problem recursively as

$$V_t(k, b, A) = \max_{c,n,k',b'} \left\{ U(c) + \beta \int_0^{\infty} V_{t+1}(k', b', A') \psi(A') \mathrm{d}A' \right\} \tag{2.6}$$

subject to the household's budget constraint

$$c + k' + b' \leq \pi + Rb + \omega \tag{2.7}$$

the expression for the profit

$$\pi = F(Ak, n) - \omega n \tag{2.8}$$

and the non-negativity and solvency constraints:

$$c \geq 0, \qquad k' \geq 0, \qquad b' \geq -h_t \tag{2.9}$$

d. Since the production function $F(\cdot)$ is CRS, it is homogeneous of degree one in (k_t^i, n_t^i). This means that for $k_t^i > 0$, the expression for the firm's profit can be written as

$$\pi_t^i = \left[F\left(A_t^i, \frac{n_t^i}{k_t^i} \right) - \omega_t \frac{n_t^i}{k_t^i} \right] k_t^i \tag{2.10}$$

As A_t^i and k_t^i are known at the time when n_t^i is chosen, optimal n_t^i/k_t^i maximizes (2.10) for any A_t^i, yielding

$$n_t^i = n(A_t^i, \omega_t) k_t^i \quad \text{and} \quad \pi_t^i = r(A_t^i, \omega_t) k_t^i \tag{2.11}$$

where

$$n(A, \omega) = \arg\max_{n \in \mathbb{R}_+} \left\{ F\left(A, \frac{n}{k} \right) - \omega \frac{n}{k} \right\}$$

and

$$r(A, \omega) = \max_{n \in \mathbb{R}_+} \left\{ F\left(A, \frac{n}{k} \right) - \omega \frac{n}{k} \right\}$$

So, as (2.11) suggests, both the individual labor demand n_t^i and the capital income π_t^i are linear functions of the firm's capital holdings k_t^i. Now, as $n(A,\omega)$ is implicitly defined by[1]

$$F_2(Ak, n(A,\omega)) = \omega$$

we can implicitly differentiate this condition with respect to ω, having

$$F_{22}(Ak, n(A,\omega))\frac{\mathrm{d}}{\mathrm{d}\omega}n(A,\omega) = 1$$

implying that

$$\frac{\mathrm{d}n}{\mathrm{d}\omega} = \frac{1}{F_{22}} < 0$$

since $F(\cdot)$ is concave. Likewise, implicit differentiation with respect to A yields

$$kF_{12}(Ak, n(A,\omega)) + F_{22}(Ak, n(A,\omega))\frac{\mathrm{d}}{\mathrm{d}A}n(A,\omega) = 0$$

meaning that[2]

$$\frac{\mathrm{d}n}{\mathrm{d}A} = -\frac{kF_{12}}{F_{22}} \geq 0$$

Finally, by the Envelope theorem, we have

$$\frac{\partial}{\partial\omega}r(A,\omega) = \frac{\partial}{\partial\omega}\left\{F\left(A, \frac{n}{k}\right) - \omega\frac{n}{k}\right\} = -\frac{n(A,\omega)}{k} < 0$$

and

$$\frac{\partial}{\partial A}r(A,\omega) = \frac{\partial}{\partial A}\left\{F\left(A, \frac{n}{k}\right) - \omega\frac{n}{k}\right\} = F_1\left(A, \frac{n}{k}\right) > 0$$

meaning that the profit π_t^i is also decreasing in ω_t and increasing in A_t^i.

e. If we define by W_t^i the individual i's *financial* wealth in period t, the household's budget constraint (2.7) reduces to[3]

$$c_t^i + k_{t+1}^i + b_{t+1}^i = W_t^i$$

and since we have demonstrated that the profits π_t^i are linear in k_t^i, the right-hand side of the above constraint can be written as

$$W_t^i = r(A_t^i, \omega_t)k_t^i + R_t b_t^i + \omega_t$$

This would allow us to eliminate the choice of optimal employment (that is : the *labor demand* schedule for agent i) in (2.6) and condition the value function $V_t(\cdot)$ on the financial wealth W only:

$$V_t(W) = \max_{c,k',b'}\left\{U(c) + \beta\int_0^\infty V_{t+1}(W')\psi(A')\mathrm{d}A'\right\} \tag{2.12}$$

subject to the budget constraint

$$c + k' + b' = W \tag{2.13}$$

[1] Subscript l denotes the partial derivative with respect to the lth argument.

[2] The reader can check that CRS assumption necessarily implies that the factors are *complementary*, i.e. the cross-derivatives of $F(\cdot)$ are non-negative.

[3] Obviously, optimality would require the budget constraint to hold with equality in each period.

the law of motion for the financial wealth

$$W' = r(A', \omega_{t+1})k' + R_{t+1}b' + \omega_{t+1} \tag{2.14}$$

and the restrictions on the choice variables: $(c, k', b') \in \mathbb{R}_+^2 \times [-h_t, \infty)$. Note that we have just shown that individual value function as well as individual wealth was independent of labor demand decision of the privately-owned firm.

Let us *guess and verify* that the value function, optimal consumption and capital holdings take the form

$$V_t(W) = U(a_t(W + h_t)) \tag{2.15a}$$

$$c_t(W) = (1 - \zeta_t)(W + h_t) \tag{2.15b}$$

$$k_{t+1}(W) = \phi_t \zeta_t (W + h_t) \tag{2.15c}$$

where a_t, ζ_t and ϕ_t are (potentially time-varying, but non-stochastic) coefficients to be determined. To interpret the above conditions, note that the sum $W + h_t$ represents the "effective" wealth of household i, ζ_t the fraction of effective wealth that is saved and ϕ_t the fraction of savings that is allocated to capital.

Plugging the expressions for consumption and capital, (2.15b) and (2.15c), into the budget constraint (2.13), we get the expression for bond holdings:

$$b_{t+1}(W) = (1 - \phi_t)\zeta_t(W + h_t) - h_t \tag{2.15d}$$

which is indeed linear in W_t, provided that $c_t(W)$ and $k_t(W)$ are. Now, plugging c from (2.13) into (2.12) and taking FOC's with respect to k' and b', we get

$$k' : \qquad U'(c) = \beta \int_0^\infty \frac{\partial V_{t+1}(W')}{\partial W'} r(A', \omega_{t+1}) \psi(A') \mathrm{d}A' \tag{2.16a}$$

$$b' : \qquad U'(c) = \beta R_{t+1} \int_0^\infty \frac{\partial V_{t+1}(W')}{\partial W'} \psi(A') \mathrm{d}A' \tag{2.16b}$$

Combining the two equations and denoting $r(A', \omega_{t+1}) \equiv r_{t+1}$ gives a nice expression we can easily interpret

$$\mathbb{E}_t \left[\frac{\partial V_{t+1}(W')}{\partial W'} r_{t+1} \right] = \mathbb{E}_t \left[\frac{\partial V_{t+1}(W')}{\partial W'} R_{t+1} \right]$$

Now, invoking the guess for $V_t(W)$, condition (2.15a) gives

$$\mathbb{E}_t \left[a_{t+1}^{1-\frac{1}{\theta}} (W' + h_{t+1})^{-\frac{1}{\theta}} (r_{t+1} - R_{t+1}) \right] = 0$$

Note that as a_t is non-stochastic, it can be factored out. Using W' from (2.14), we can write

$$W' = r_{t+1}k' + R_{t+1}b' + \omega_{t+1} = [\phi_t r_{t+1} + (1 - \phi_t) R_{t+1}] \zeta_t (W + h_t) - h_{t+1}$$

Finally, we can replace $W' + h_{t+1}$ using above and factor out the term known at date t and we obtain

$$\mathbb{E}_t\left[\left[\phi_t r_{t+1} + (1-\phi_t)R_{t+1}\right]^{-\frac{1}{\theta}}(r_{t+1} - R_{t+1})\right] = 0$$

This implicitly defines $\phi_t = \phi(R_{t+1}, \omega_{t+1})$.

Next, we use the envelope condition $V'(W) = U'(c)$. Noting that $U'(c) = c^{-1/\theta}$ and $V'(W) = a_t U'(a_t(W + h_t)) = a_t(a_t(W + h_t))^{-1/\theta}$, and using the guess for c_t we get:

$$a_t^{1-\frac{1}{\theta}} = (1-\zeta_t)^{-\frac{1}{\theta}}$$

Multiplying (2.16b) and (2.16a) by $(1-\phi_t)$ and ϕ_t respectively, summing up and replacing $W' + h_{t+1}$ as we did above, one can get the recursive definition of ζ_t

$$(1-\zeta_t)^{-1} = 1 + \beta^\theta \rho_t^\theta (1-\zeta_{t+1})^{-1}$$

where $\rho_t = \rho(\omega_{t+1}, R_{t+1})$. One can forward iterate this expression to get the following expression

$$\zeta_t = \left\{1 + \left[\sum_{\tau=t}^{\infty}\prod_{j=t}^{\tau}\beta^\theta \rho_j^{\theta-1}\right]^{-1}\right\}^{-1}$$

Hence, every parameter can be defined. One must now verify that the guess satisfy all equilibrium conditions which is a bit tedious but can be done...

Why was is interesting to show that relevant variables are *linear* in wealth? Angeletos (2007) explains this very well :
“A common difficulty with incomplete-market models is that the wealth distribution an infinite-dimensional object is a relevant state variable for aggregate dynamics. This is not the case in this model. As long as private firms operate under a neoclassical CRS technology (e.g., a Cobb- Douglas technology), the individual agent faces risky, but linear, returns to his own investment. Together with the homotheticity of preferences (CRRA/CEIS), this ensures that, for any given sequence of prices, optimal decision rules are linear in individual wealth. By implication, aggregate quantities and prices in equilibrium are invariant to the wealth distribution. This avoids the “curse of dimensionality” and permits characterizing the equilibrium with a low-dimensional, closed-form dynamic system. The analysis of the steady state is then particularly tractable.”

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The following pages give a review of E?ciency in General Equilibrium with Incomplete Markets. The solution to problem 3 is in the subsection 2.2.3 Example 2: Three Periods, One Good.

Efficiency in General Equilibrium with Incomplete Markets

1 The Two-Period GEI Model

1.1 The Model

A two-period, C-good, GEI economy is defined by $\left(S, \left(u^h, e^h\right)_{h\in H}, A\right)$, where $S = \{0, 1, \ldots, S\}$ is the set of nodes, $u^h : \mathbb{R}^{(S+1)C} \to \mathbb{R}$ is weakly monotonic, $e^h \in \mathbb{R}^{(S+1)C}$ for all $h \in H$, and $A \in \mathbb{R}^{(SC)\times J}$ is the matrix of assets and has rank J. A GEI equilibrium $\left(p, \pi, \left(x^h, \theta^h\right)_{h\in H}\right)$, where $p \in \mathbb{R}^{(S+1)C}$, $\pi \in \mathbb{R}^J$, and for all $h \in H$, $x^h \in \mathbb{R}^{(S+1)C}$, $\theta^h \in \mathbb{R}^J$, satisfies:

1. $\sum_{h\in H} \left(x^h - e^h\right) = 0$
2. $\sum_{h\in H} \theta^h = 0$
3. $\forall h \in H, \left(x^h, \theta^h\right) \in B^h(p, \pi)$
4. $(x, \theta) \in B^h(p, \pi) \Rightarrow \forall h \in H, u^h(x) \leq u^h(x^h)$

where

$$B^h(p,\pi) \equiv \Big\{(x,\theta) \in \mathbb{R}^{(S+1)C} \times \mathbb{R}^J : \quad p_0 \cdot (x_0 - e_0) + \pi \cdot \theta \leq 0 \\ \text{and } \forall s \geq 1, \quad p_s \cdot (x_s - e_s) \leq \sum_{j\in J} p_s \cdot A_{sj}\theta_j \Big\}$$

For any vector $y = (y_0, y_1, \ldots, y_S) \in \mathbb{R}^{(S+1)C}$, we note $y = (y_0, \tilde{y})$, with $\tilde{y} \in \mathbb{R}^{SC}$. For any $\tilde{p} \in \mathbb{R}^{SC}$ and $\tilde{y} \in \mathbb{R}^{SC}$, we note $\tilde{p} \circ \tilde{y} = (p_1 \cdot y_1, \ldots, p_S \cdot y_S)' \in \mathbb{R}^S$ the money value of the bundle y in every state. For any $\tilde{p} \in \mathbb{R}^{SC}$ and $A \in \mathbb{R}^{(SC)\times J}$, we note $\tilde{p} \circ A = (\tilde{p} \circ A^1, \ldots, \tilde{p} \circ A^J) \in \mathbb{R}^{S\times J}$.

1.2 No Arbitrage

Given the asset payoff matrix $A \in \mathbb{R}^{(SC)\times J}$ and GEI prices (p, π), we say that the triple (p, π, A) does not allow for arbitrage iff there is no solution $\theta \in \mathbb{R}^J$ solving the following system $\mathscr{S}$

$$\pi \cdot \theta \leq 0$$
$$\forall s \geq 1, ([\tilde{p} \circ A]\theta)_s \geq 0$$

with one strict inequality, where $([\tilde{p} \circ A]\theta)_s = \sum_{j\in J} p_s \cdot A_{sj}\theta_j = p_s \cdot A_s\theta$.

Theorem 1.1 (No Arbitrage). *Let* $\left(p, \pi, \left(x^h, \theta^h\right)_{h \in H}\right)$ *be a GEI equilibrium for a GEI economy* $\left(S, \left(u^h, e^h\right)_{h \in H}, A\right)$ *satisfying weak monotonicity. Then* (p, π, A) *does not allow for arbitrage.*

Proof. If there was an arbitrage θ, then any agent h would do better to replace θ^h by $\theta^h + \theta$, consume strictly more of every good in some state s (corresponding to the strict inequality in $\mathscr{S}$), without losing any consumption in any other state. This contradicts weak monotonicity. □

1.3 Personal Stochastic Discounting

Let $\left(p, \pi, \left(x^h, \theta^h\right)_{h \in H}\right)$ be a GEI equilibrium for a GEI economy $\left(S, \left(u^h, e^h\right)_{h \in H}, A\right)$. Suppose that there exists an h such that u^h is differentiable and $\forall s \in S$, $p_s \cdot x^h_s > 0$, i.e. $\forall s \in S, \exists c(s) \in C$ s.t. $x^h_{s,c(s)} > 0$ and $\forall s \in S, p_{s,c(s)} > 0$. Define the marginal utility of money to agent h in state $s \geq 1$, divided by the marginal utility of money in state 0 by

$$\mu^h_s \equiv \left(\frac{1}{p_{s,c(s)}} \frac{\partial u^h(x^h)}{\partial x^h_{s,c(s)}}\right) \left(\frac{1}{p_{0,c(0)}} \frac{\partial u^h(x^h)}{\partial x^h_{0,c(0)}}\right)^{-1}$$

Note that μ^h_s is well-defined in every state and does not depend on the choice of $c(s)$ such that $x_{s,c(s)} > 0$ (proof: write the FOC for $c(s)$ and $c'(s)$).

Theorem 1.2 (Marginal Utility Pricing). *For any* $j \in J$*, the price of asset* j *is given by*

$$\pi_j = \sum_{s=1}^{S} p_s \cdot A_{sj} \mu^h_s \quad \textit{i.e.} \quad \pi' = \left(\mu^h\right)' (\tilde{p} \circ A)$$

Proof. The FOC for $c(s)$, $s \geq 0$, writes $\partial u^h(x^h)/\partial x^h_{s,c(s)} = \lambda_s p_{s,c(s)}$. The FOC for θ_j writes $-\lambda_0 \pi_j + \sum_s \lambda_s p_s \cdot A_{sj} = 0$. Combining these equations gives the result. □

This is the fundamental theorem of asset pricing. It says that the price of an asset is equal to the marginal utilities of the dividends it pays. The personal stochastic discounts also reveal that stochastic discounting is far from unique, when markets are incomplete.

1.4 Stochastic Discounting

Theorem 1.3 (Stochastic Discounting). (p, π, A) *does not allow for arbitrage iff.*

$$\exists (\mu_1, \ldots, \mu_S) \gg 0 \textit{ s.t. } \forall j \in J,\ \pi_j = \sum_{s=1}^{S} p_s \cdot A_{sj} \mu_s \quad \textit{i.e.} \quad \pi' = \mu' (\tilde{p} \circ A)$$

Proof. It relies on Farkas' lemma, an arithmetic version of the separating hyperplane theorem.

Lemma 1.4 (Farkas). *The system* $Mx \geq 0$*, with one strict inequality (say line s), where* $M \in \mathbb{R}^{n \times p}$*, has no solution* $x \in \mathbb{R}^p$ *iff.* $\exists \mu^s = \left(\mu^s_0, \ldots, \mu^s_{(n-1)}\right)' \geq 0$ *s.t.* $(\mu^s)' M = (0, \ldots, 0)$ *and* $\mu^s_s > 0$.

Let $n = S + 1$, $p = J$, $m_{0j} = -\pi_j$ and $\forall s \geq 1$, $m_{sj} = p_s \cdot A_{sj}$. Because there is no arbitrage, we know that for any $s \geq 0$, the system $\mathscr{S}$ with a strict inequality in line s has no solution. Thus $\exists \mu^s \geq 0$ s.t. $\mu_s^s > 0$ and $\mu_0^s(-\pi)' + \left(\tilde{\mu^s}\right)' (\tilde{p} \circ A) = 0$. Let $\mu \equiv (\mu_1, \ldots, \mu_S) \equiv \left(\sum_{s=0}^S \mu_1^s / \sum_{s=0}^S \mu_0^s, \ldots, \sum_{s=0}^S \mu_S^s / \sum_{s=0}^S \mu_0^s\right) \gg 0$. Then $(-\pi)' + (\tilde{\mu})' (\tilde{p} \circ A) = 0$. □

This theorem says that assets are priced as if everyone agreed on discounts or weights μ_s for each state s, so that the price of every asset could be computed by taking the weighted average of (stochastically discounted) payoffs. If utilities are smooth and agents are consuming a strictly positive amount of some good in every state, then the stochastic discount theorem is simply an expression of the fact that asset prices are given by marginal utilities.

Theorem 1.5 (Martingale Pricing). *Let $\left(p, \pi, \left(x^h, \theta^h\right)_{h \in H}\right)$ be a GEI equilibrium for a GEI economy $\left(S, \left(u^h, e^h\right)_{h \in H}, A\right)$ satisfying weak state-by-state monotonicity. Then there exists a probability $\nu \gg 0$ on S, and $r > -1$, s.t. for all $j \in J$,*

$$\pi_j = \frac{1}{1+r} \mathbb{E}_\nu \left[p_s \cdot A_{sj}\right] \equiv \frac{1}{1+r} \sum_{s=1}^S p_s \cdot A_{sj} \nu_s$$

Proof. Let $\nu_s = \mu_s / \left(\sum_{\tau=1}^S \mu_\tau\right)$ and $1/(1+r) = \sum_{\tau=1}^S \mu_\tau$. □

Theorem 1.6 (Relative Martingale Pricing). *Furthermore, there exists a probability $\lambda \gg 0$ on S s.t. if $\forall s \in S, p_s \cdot A_{sJ} > 0$, then for all $j \in J$,*

$$\frac{\pi_j}{\pi_J} = \sum_{s=1}^S \frac{p_s \cdot A_{sj}}{p_s \cdot A_{sJ}} \lambda_s$$

Proof. Let $\lambda_s = p_s \cdot A_{sJ} \mu_s / \pi_J$, then we have $\sum_s \lambda_s = 1$. □

This is the "zenith of classical finance ideology". However, the probabilities ν and λ have nothing to do with the real world!

2 Pareto Optimality and Constrained Efficiency

A corollary of what has been said up to now is the following. Let $\left(p, \pi, \left(x^h, \theta^h\right)_{h \in H}\right)$ be a GEI equilibrium for a smooth two-period GEI economy $\left(S, \left(u^h, e^h\right)_{h \in H}, A\right)$. We know that for all h, $\pi' = \left(\mu^h\right)' (\tilde{p} \circ A)$. Thus $\forall \tilde{z} \in \text{Span}\,(\tilde{p} \circ A)$, $\forall (h, i) \in H$, $\mu^h \cdot \tilde{z} = \mu^i \cdot \tilde{z}$. In other words, for all $h \in H$,

$$\text{Proj}\big|_{\text{Span}(\tilde{p} \circ A)} \mu^h = \mu \quad \text{where} \quad \forall s \geq 1,\ \mu_s^h \equiv \left(\frac{1}{p_{s,c(s)}} \frac{\partial u^h(x^h)}{\partial x^h_{s,c(s)}}\right) \left(\frac{1}{p_{0,c(0)}} \frac{\partial u^h(x^h)}{\partial x^h_{0,c(0)}}\right)^{-1}$$

Therefore μ is the projection of individual μ^h onto the span of $\tilde{p} \circ A$, for every h.

2.1 Pareto Efficiency

Lemma 2.1. *Let* $\left(p, \pi, \left(x^h, \theta^h\right)_{h\in H}\right)$ *be a GEI equilibrium for a smooth two-period GEI economy* $\left(S, \left(u^h, e^h\right)_{h\in H}, A\right)$*. Then* $\left(x^h\right)_{h\in H}$ *is Pareto Optimal if and only if*

$$\forall h, h' \in H, \quad \mu^h = \mu^{h'}$$

An allocation is Pareto optimal iff. marginal utilities of income in the different states are collinear across agents, when consumption at time 1 is positive. The μ^h can be very different from one another and still be projected into the same μ because the span of the assets is so small. When the span of the assets is $\mathbb{R}^S$, then the projection of each μ^h is μ^h itself, so that all the μ^h's have to be collinear, which in turn implies optimality. When there are fewer assets than there are states, on the contrary, then there can be different μ^h for different individuals, although they project down to the same thing. In fact, we have the following theorem:

Theorem 2.2 (Pareto Inefficiency)**.** *Let* $\left(u^h\right)_{h\in H}$ *be smooth. Suppose* $J < S$*,* A *fixed. Then for almost all assignments of the endowments* $\left(e^h\right)_{h\in H}$*, the resulting economy* $\left(S, \left(u^h, e^h\right)_{h\in H}, A\right)$ *has no Pareto efficient equilibria.*

Proof. One-good ($C = 1$) Case. To be done. □

2.2 Constrained Efficiency

2.2.1 Constrained Efficiency in the Two-period, One-Good Case

Definition 2.1. *Given the one-good GEI economy* $\left(S, \left(u^h, e^h\right)_{h\in H}, A\right)$*, the allocation* $\left(x^h\right)_{h\in H}$ *is constrained efficient if and only if by using only existing assets no other feasible allocation is Pareto better, i.e., if* $\left(\hat{x}^h, \hat{\theta}^h\right)_{h\in H}$ *satisfies* $\sum_h \hat{x}^h = \sum_h \hat{e}^h$*,* $\sum_h \hat{\theta}^h = 0$*,* $\tilde{\hat{x}}^h = \tilde{\hat{e}}^h + A\hat{\theta}^h$ $\forall h$*, then we cannot have* $u^h(\hat{x}^h) > u^h(x^h)$ $\forall h$*.*

The allocation is constrained inefficient if the government can intervene at time 0 without adding new markets and make everybody better off.

Theorem 2.3. *Let* $\left(p, \pi, \left(x^h, \theta^h\right)_{h\in H}\right)$ *be a GEI equilibrium for a two-period, one-good GEI economy* $\left(S, \left(u^h, e^h\right)_{h\in H}, A\right)$*. Then* $\left(x^h\right)_{h\in H}$ *is constrained efficient.*

Proof. Suppose $u^h(\hat{x}^h) > u^h(x^h)$ $\forall h$ and $\tilde{\hat{x}}^h = \tilde{\hat{e}}^h + A\hat{\theta}^h$. Then it must be that $p_0\hat{x}_0^h + \pi \cdot \hat{\theta}^h > p_0 e_0^h$. Summing over h implies $p_0 \sum_h \hat{x}_0^h > p_0 \sum_h e_0^h$, contradiction. □

2.2.2 Example 1: Two Periods, Two Goods

The economy is described by

$$e^A = \left(e_0^A, \left(e_{11}^A, e_{12}^A\right), \left(e_{21}^A, e_{22}^A\right)\right) = (1,(3,3),(3,0))$$
$$e^B = \left(e_0^B, \left(e_{11}^B, e_{12}^B\right), \left(e_{21}^B, e_{22}^B\right)\right) = (1,(2,1),(1,5))$$
$$u^A\left(x_0,\left(x_{11},x_{12}\right),\left(x_{21},x_{22}\right)\right) = x_0 + \frac{1}{2}\left\{\frac{2}{3}\log\left(x_{11}\right)+\frac{1}{3}\log\left(x_{12}\right)\right\} + \frac{1}{2}\left\{\frac{2}{3}\log\left(x_{21}\right)+\frac{1}{3}\log\left(x_{22}\right)\right\}$$
$$u^B\left(x_0,\left(x_{11},x_{12}\right),\left(x_{21},x_{22}\right)\right) = x_0 + \frac{1}{2}\left\{\frac{1}{3}\log\left(x_{11}\right)+\frac{2}{3}\log\left(x_{12}\right)\right\} + \frac{1}{2}\left\{\frac{1}{3}\log\left(x_{21}\right)+\frac{2}{3}\log\left(x_{22}\right)\right\}$$

There is a unique asset, which pays one unit of each good in each state: $A = (1,1,1,1)'$.

Equilibrium.

$$\begin{aligned}\mathscr{E} &= \left(\left\{p_0,\left(p_{11},p_{12}\right),\left(p_{21},p_{22}\right),\pi\right\},\left\{x_0^A,\left(x_{11}^A,x_{12}^A\right),\left(x_{21}^A,x_{22}^A\right),\theta^A\right\},\left\{x_0^B,\left(x_{11}^B,x_{12}^B\right),\left(x_{21}^B,x_{22}^B\right),\theta^B\right\}\right)\\ &= \left(\{1,(1,1),(1,1),1/2\},\{1,(4,2),(2,1),0\},\{1,(1,2),(2,4),0\}\right)\end{aligned}$$

It is easy to verify that the marginal utility of good one is the same as that of good 2 for each agent in each state, and that each agent is on his budget set. Finally, the marginal utility of the asset is $\mu_\theta^A = \frac{1}{2}\{(2/3)(1/4)+(1/3)(1/2)\} + \frac{1}{2}\{(2/3)(1/2)+(1/3)(1/1)\} = 1/2 = \pi\left(\mu_0^A/p_0\right)$. Similarly, $\mu_\theta^B = \pi\left(\mu_0^B/p_0\right)$. Hence we have an equilibrium.

Inefficiency. The marginal utility of money in each state for agent A and agent B is

$$\left(\mu_1^A,\mu_2^A\right) = \left(\frac{1}{2}\left\{\frac{2}{3}\cdot\frac{1}{4}\right\},\frac{1}{2}\left\{\frac{2}{3}\cdot\frac{1}{2}\right\}\right) = \left(\frac{1}{12},\frac{1}{6}\right) \quad \text{and} \quad \left(\mu_1^A,\mu_2^A\right) = \left(\frac{1}{6},\frac{1}{12}\right)$$

Since these are not equal, the equilibrium is *not* Pareto efficient. Intuitively, agents would like to trade an asset in order to transfer wealth from one state to another. But there exists a unique asset which pays the same in both states, and hence does not facilitate such a transfer.

Constrained Inefficiency. Suppose that the planner forces agent A (resp., B) to buy (resp., sell) a little more of the asset, and let markets clear in period 1. Then A (resp., B) will be richer (resp., poorer) in both states in the future. But A's marginal propensity to consume good 1 is larger than B's (i.e. $2/3 > 1/3$). Thus the price of good 1 will increase relative to the price of good 2 in both states. Since A is a buyer of good 1 in state 1, this price change makes him worse off, and B better off, in state 1. Similarly, A is better off and B worse off in state 2. On balance, both will be better off. The government has effectively provided insurance by intervening at time 0 on the existing markets[1].

Formally, let A buy θ units of the asset from B. Holding $p_{12} = 1$, the market clearing price p_{11} of good 1 in state 1 solves $\frac{2}{3p_{11}}\{(3+\theta)p_{11}+(3+\theta)\} + \frac{1}{3p_{11}}\{(2-\theta)p_{11}+(1-\theta)\} = 5$. Similarly,

[1]Intuitively, the seller of good 1 is always the poor guy, hence the policy helps the person who is in the poor state.

in state 2, holding $p_{22} = 1$, we have $\frac{2}{3p_{21}}\{(3+\theta)p_{21} + (0+\theta)\} + \frac{1}{3p_{21}}\{(1-\theta)p_{11} + (5-\theta)\} = 4$. Hence

$$\left.\frac{dp_{11}}{d\theta}\right|_{\theta=0} = \frac{2}{7} \quad \text{and} \quad \left.\frac{dp_{21}}{d\theta}\right|_{\theta=0} = \frac{2}{5}$$

Now, by the envelope theorem, an infinitesimal increase in θ will cause utilities to change by the change in the cost of buying the same commodity bundle, times the marginal utility of income. Formally, let

$$\begin{aligned} V = \max u\left(x_0, x_{11}, x_{12}, x_{21}, x_{22}\right) & \\ \text{s.t.} \quad & x_0 + \pi\theta = e_0 \\ \text{and} \quad & p_{11}\left(x_{11} - e_{11}\right) + \left(x_{12} - e_{12}\right) = p_{11}\theta + \theta \\ \text{and} \quad & p_{21}\left(x_{21} - e_{21}\right) + \left(x_{22} - e_{22}\right) = p_{21}\theta + \theta \end{aligned}$$

Applying the envelope theorem yields

$$\begin{aligned} \left.\frac{dV}{d\theta}\right|_{\mathscr{E}} &= \left\{ -\lambda_0\pi + \lambda_1\left(p_{11}+1\right) + \lambda_2\left(p_{21}+1\right) \right\} + \left\{ -\lambda_1 \frac{dp_{11}}{d\theta}\left(x_{11}-e_{11}\right) - \lambda_2 \frac{dp_{21}}{d\theta}\left(x_{21}-e_{21}\right) \right\} \\ &= \left\{ -\frac{\partial u(x)}{\partial x_0}\pi + \frac{1}{p_{11}}\frac{\partial u(x)}{\partial x_{11}}\left(p_{11}+1\right) + \frac{1}{p_{21}}\frac{\partial u(x)}{\partial x_{21}}\left(p_{21}+1\right) \right\} \\ &\qquad + \left\{ -\frac{1}{p_{11}}\frac{\partial u(x)}{\partial x_{11}}\frac{dp_{11}}{d\theta}\left(x_{11}-e_{11}\right) - \frac{1}{p_{21}}\frac{\partial u(x)}{\partial x_{21}}\frac{dp_{21}}{d\theta}\left(x_{21}-e_{21}\right) \right\} \\ &= -\left\{ \frac{1}{p_{11}}\frac{\partial u(x)}{\partial x_{11}}\frac{dp_{11}}{d\theta}\left(x_{11}-e_{11}\right) + \frac{1}{p_{21}}\frac{\partial u(x)}{\partial x_{21}}\frac{dp_{21}}{d\theta}\left(x_{21}-e_{21}\right) \right\} \end{aligned}$$

If relative prices did not change in period 1, forcing A to buy a little more of the asset from B would hardly matter, since from the first order conditions the gain in consumption in period 1 by agent A is almost exactly offset by the loss in period 0, and similarly for agent B (formally, the term $-\lambda_0\pi + \lambda_1\left(p_{11}+1\right) + \lambda_2\left(p_{21}+1\right)$ above is equal to 0). But prices do change, and we obtain

$$\begin{aligned} \left.\frac{du^A}{d\theta}\right|_{\mathscr{E}} &= \frac{1}{2}\left\{\frac{2}{3}\cdot\frac{1}{4}\right\}\frac{2}{7}(-1) + \frac{1}{2}\left\{\frac{2}{3}\cdot\frac{1}{2}\right\}\frac{2}{5}(1) > 0 \\ \left.\frac{du^B}{d\theta}\right|_{\mathscr{E}} &= \frac{1}{2}\left\{\frac{1}{3}\cdot\frac{1}{1}\right\}\frac{2}{7}(1) + \frac{1}{2}\left\{\frac{1}{3}\cdot\frac{1}{2}\right\}\frac{2}{5}(-1) > 0 \end{aligned}$$

It can robustly be the case that one dV is positive and one is negative. Then if the sum is positive, increase θ and transfer from one to the other by changing e_0's. If sum is negative, decrease θ and transfer from the winner to the loser by changing e_0's.

2.2.3 Example 2: Three Periods, One Good

The economy is described by

$$e^A = \left(e_0^A, \left(e_{u1}^A, e_{u2}^A\right), \left(e_{d1}^A, e_{d2}^A\right)\right) = (1, (0,0), (0,0))$$
$$e^B = \left(e_0^B, \left(e_{u1}^B, e_{u2}^B\right), \left(e_{d1}^B, e_{d2}^B\right)\right) = (1, (0,0), (0,0))$$
$$u^A\left(x_0, \left(x_{u1}, x_{u2}\right), \left(x_{d1}, x_{d2}\right)\right) = \frac{1}{2}\log\left(x_{u1}\right) + \frac{1}{2}\beta\log\left(x_{d2}\right)$$
$$u^B\left(x_0, \left(x_{u1}, x_{u2}\right), \left(x_{d1}, x_{d2}\right)\right) = \frac{1}{2}\log\left(x_{d1}\right) + \frac{1}{2}\beta\log\left(x_{u2}\right)$$

There is a ("short-run") firm which produces 1 unit of good in states $u1$ and $d1$ using 1 unit of good in state 0, and a ("long-run") firm which produces $R > 1$ units of good in states $u2$ and $d2$ using 1 unit of good in state 0. A owns a share $\alpha^{SR} \in [0,1]$ of the short-run firm, and a share $\alpha^{LR} \in [0,1]$ of the long-run firm.

Arrow-Debreu Equilibrium. Let q_i be the price of the consumption good in state i, $\left(x_i^h\right)_{h\in H}$ be the quantities consumed, and $\left(y_i^f\right)_{f\in F}$ be the quantities produced. In equilibrium, both firms produce, and make zero profit since they have constant returns to scale. Hence $q_{u1} + q_{d1} = 1$ and $q_{u2} + q_{d2} = 1/R$. By symmetry, we obtain

$$\left(q_0, \left(q_{u1}, q_{d1}\right), \left(q_{u2}, q_{d2}\right)\right) = \left(1, \left(\frac{1}{2}, \frac{1}{2R}\right), \left(\frac{1}{2}, \frac{1}{2R}\right)\right)$$

A solves max $\frac{1}{2}\log\left(x_{u1}\right) + \frac{1}{2}\beta\log\left(x_{d2}\right)$ s.t. $\frac{1}{2}x_{u1} + \frac{1}{2R}x_{d2}$. We find

$$\begin{aligned}&\left(\left\{x_0^A, \left(x_{u1}^A, x_{d1}^A\right), \left(x_{u2}^A, x_{d2}^A\right)\right\}, \left\{x_0^B, \left(x_{u1}^B, x_{d1}^B\right), \left(x_{u2}^B, x_{d2}^B\right)\right\}\right)\\ &\qquad = \left(\left\{0, \left(\frac{2}{1+\beta}, 0\right), \left(0, \frac{2R\beta}{1+\beta}\right)\right\}, \left\{0, \left(\frac{2}{1+\beta}, 0\right), \left(0, \frac{2R\beta}{1+\beta}\right)\right\}\right)\end{aligned}$$
$$\left(\left\{y_0^{SR}, y_{u1}^{SR}, y_{d1}^{SR}\right\}, \left\{y_0^{LR}, y_{u2}^{LR}, y_{d2}^{LR}\right\}\right) = \left(\left\{\frac{-2}{1+\beta}, \frac{2}{1+\beta}, \frac{2}{1+\beta}\right\}, \left\{\frac{-2}{1+\beta}, \frac{2R\beta}{1+\beta}, \frac{2R\beta}{1+\beta}\right\}\right)$$

Incomplete Markets Equilibrium. There are the same production possibilities, but there is no insurance. Suppose that there is a short bond at each state which pays 1 in every immediate successor, and there is a long bond in period 0 which pays 1 at each terminal node. We denote their prices by π_0^{SR}, π_{u1}^{SR}, π_{d1}^{SR}, π_0^{LR}. Let the price of the consumption good be 1 in every state.

Suppose that $\pi_0^{SR} > 1$. Then A would have an arbitrage opportunity. Indeed, A could hold $\theta_0^{SR} < 0$ units of the short-run bond (hence he promises to deliver $|\theta_0^{SR}|$ units tomorrow) and invest $\pi_0^{SR}|\theta_0^{SR}|$ in the short-run production; he would thus make a strictly positive profit $\pi_0^{SR}|\theta_0^{SR}| - |\theta_0^{SR}| > 0$ tomorrow. Suppose that $\pi_0^{SR} < 1$. Then no production would take place, and both A and B would want to buy, and no one would sell, the bond, hence this cannot be an equilibrium either. Thus $\pi_0^{SR} = 1$. Similarly, $\pi_0^{LR} = 1/R$. Finally, let us assume that $\pi_{u1}^{SR} = \pi_{d1}^{SR}$

in equilibrium. Then the same reasoning (no arbitrage in state $u1$) shows that we necessarily have $\pi^{SR}_{u1} = 1/R$.

A's budget constraints write $\{\theta^{SR}_0 + y^{SR}_0\} + \{(1/R)\theta^{LR}_0 + y^{LR}_0\} = 1$ in period 0, $x_{u1} + (1/R)\theta^{SR}_{u1} = \{\theta^{SR}_0 + y^{SR}_0\}$ in state $u1$, $0 = \theta^{SR}_{u1} + \{\theta^{LR}_0 + Ry^{LR}_0\}$ in state $u2$, $(1/R)\theta^{SR}_{d1} = \{\theta^{SR}_0 + y^{SR}_0\}$ in state $d1$, and $x_{d2} = \theta^{SR}_{d1} + \{\theta^{LR}_0 + Ry^{LR}_0\}$ in state $d2$. Thus $x^A_{u1} = 1$, $x^A_{d2} = R$, , and by symmetry, $x^B_{d1} = 1$, $x^B_{u2} = R$.

Goods market clearing in all states then requires that the production be $y^{SR} = (-1, 1, 1, 0, 0)$ and $y^{LR} = (-1, 0, 0, R, R)$. Suppose that $\alpha^{SR} = \alpha^{LR} = 1$, i.e. A owns both firms. Then $y^{A,SR}_0 = -1$ and $y^{A,LR}_0 = -1$. So since $x^B_{d1} = 1$, we have $\theta^{B,SR}_0 = 1$ and by bond market clearing, $\theta^{A,SR}_0 = -1$ (this is consistent with $y^A_0 = -2$, i.e. A borrows B's endowment at time 0 and invests it in the technologies). In state $u1$, A has net endowment 0, because he gets output $+1$ but owes debt -1, and he will have $+R$ in state $u2$, whereas B has $+1$ in $u1$ (that A owes him) and will have 0 in $u2$. So $\theta^{A,SR}_{u1} = -R$ and $\theta^{B,SR}_{u1} = R$, i.e. B buys A's future consumption at price $\pi^{SR}_{u1} = 1/R$. A similar reasoning shows that in state $d1$, we have $\theta^{A,SR}_{d1} = \theta^{B,SR}_{d1} = 0$. Finally, $\theta^{A,LR}_0 = \theta^{B,LR}_0 = 0$. Suppose instead that $\alpha^{SR} = \alpha^{LR} = 1/2$. Then we have $\theta^{A,SR}_0 = \theta^{B,SR}_0 = 0$, $\theta^{A,LR}_0 = \theta^{B,LR}_0 = 0$, and $\theta^{A,SR}_{u1} = \theta^{B,SR}_{d1} = -R/2$, $\theta^{A,SR}_{d1} = \theta^{B,SR}_{u1} = R/2$.

Pareto Efficiency The competitive equilibrium with incomplete markets is inefficient. In particular, x^A_{u1} is too low, since $1 < 2/(1+\beta)$. Note however that the interest rates are the same as in the Arrow-Debreu equilibrium. The Pareto improving time-0 intervention will move some prices *away* from the corresponding Arrow-Debreu prices. This is a typical feature of "second best".

Constrained Efficiency How can we make everyone better off? Both A and B have a common interest in helping their *impatient* self, since A in state $u1$ and B in state $u2$ have a larger marginal utility. We will thus redistribute wealth toward the impatient guy by decreasing the interest rate, which will be achieved by forcing them to do more short-run production. This will make both A and B better off, since their impatient self always borrows. Decreasing the interest rate makes it cheaper to borrow.

Consider the case where $\alpha^{SR} = \alpha^{LR} = 1/2$. In this case, A and B invest $-y^{A,SR}_0 = -y^{B,SR}_0 = 1/2$ in short-run production. Suppose that the government imposes $-y^{A,SR}_0, -y^{B,SR}_0 \geq \bar{s}$, where $\bar{s} \in (1/2, 1]$. This implies $-y^{A,LR}_0, -y^{B,LR}_0 \leq 1 - \bar{s}$. What will the new interest-rate R' be? A's budget constraints in each state now write $y^{SR}_0 + y^{LR}_0 = 1$ in period 0, $x_{u1} + (1/R')\theta^{SR}_{u1} = y^{SR}_0$ in state $u1$, $0 = \theta^{SR}_{u1} + Ry^{LR}_0$ in state $u2$, $(1/R')\theta^{SR}_{d1} = y^{SR}_0$ in state $d1$, and $x_{d2} = \theta^{SR}_{d1} + Ry^{LR}_0$ in state $d2$. Moreover, we will have $y^{SR}_0 = \bar{s} = 1 - y^{LR}_0$. Therefore we obtain $\theta^{A,SR}_{u1} = -R(1-\bar{s})$, $\theta^{A,SR}_{d1} = R'\bar{s}$, $x^A_{u1} = R/R' + (1 - R/R')\bar{s}$. Now, bond market clearing in state $u1$ imposes $\theta^{A,SR}_{u1} + \theta^{B,SR}_{u1} = 0$, hence

$$R' = \frac{1-\bar{s}}{\bar{s}} R < R \quad \text{and} \quad x^A_{u1} = 2\bar{s} > 1$$

Conclusion: We can replicate the Arrow-Debreu equilibrium by taking $\bar{s} = 1/(1+\beta) > 1/2$.

2.2.4 A General Theorem

Theorem 2.4 (Constrained Ineffiency). *Fix the smooth utilities* $\left(\bar{u}^h\right)_{h\in H}$. *Let* $u^h_{\lambda^h,Q^h}(x) = \bar{u}^h(x) + \lambda^h \cdot x - \frac{1}{2}x'Q^h x$, *where* Q^h *is a nonnegative definite matrix. (Restrict the quadratic perturbation* (λ^h, Q^h) *to a small open set near 0.) Fix* $H \geq 2$, $J < S - H$, *in the economy* $\left(S, \left(\bar{u}^h\right)_{h\in H}, A\right)$. *Then for almost all* $\left(Q^h, \lambda^h, e^h\right)_{h\in H}$, *the resulting economy* $\left(S, \left(u^h_{\lambda^h,Q^h}, e^h\right)_{h\in H}, A\right)$ *has a finite number of equilibria, all of which are constrained inefficient.*

Proof. Cf. Geanokoplos and Polemarchakis (1986). □

That is, there is a reallocation of assets and goods in period 0 such that when spot markets reclear in period 1 in every state, all agents will be strictly better off.

Scope for government intervention? Markets are not good but information pb (very different argument against gvt than "efficient market hypothesis").