TSE Master 2 — Macroeconomics I
Problem Set 3

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1 Decentralized Ramsey Model with Labor and Capital Tax

1. In this economy, the households make investment. The resource allocation is:

$$C_t + K_{t+1} - (1-\delta)K_t = Y_t$$

2. The maximization problem of household is:

$$\max_{\{C_t,K_t\}_{t=0}^{+\infty}} \quad \sum_{t=0}^{+\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$

$$\text{s.t.} \quad C_t + K_{t+1} - (1-\delta)K_t = (1-\tau^L)w_t L_t + (1-\tau^K)R_t K_t + T_t \tag{2}$$

$$K_t \geq 0, \quad K_0 \text{ given} \tag{3}$$

The Bellman equation writes as:

$$V(K_t) = \max_{C_t}\{\frac{C_t^{1-\gamma}}{1-\gamma} + \beta V(K_{t+1})\}$$

The optimal conditions are:

$$C_t^{-\gamma} = \beta V'(K_{t+1}) \tag{4}$$

$$V'(K_t) = \beta V'(K_{t+1})(1-\delta+R_t(1-\tau^K)) \tag{5}$$

Combine equation (4) and (5), we have the Euler Equation:

$$\frac{C_{t+1}}{C_t} = \left[\beta(1-\delta+R_t(1-\tau^K))\right]^{\frac{1}{\gamma}} \tag{6}$$

3. If there exists steady state, we have the left hand side of equation (6) is 1, therefore, we have:

$$\beta[(1-\delta+R_t(1-\tau^K)] = 1$$

*Thanks to Alban for providing solutions from previous years. Please let us know if you find typos or errors: fei.yue@hotmail.com or lan.lansl@yahoo.fr

Equally,

$$\frac{\partial F(K_t, L_t)}{\partial K_t} = \frac{\frac{1}{\beta} + \delta - 1}{1 - \tau^K} \tag{7}$$

This is the curve which captures the dynamics of consumption. Then we turn to the LOM of capital, which captures the dynamics of capital. The LOM is capital is:

$$K_{t+1} = F(K_t, L_t) - C_t + K_t(1 - \delta)$$

Let $K_{t+1} = K_t$, then we have:

$$C_t = F(K_t, L_t) - \delta K_t \tag{8}$$

Combine (7) and (8), then we are able to characterize the steady state.

2 Razin and Sadka (1995)

This problem is taken from (Ljungqvis tSargent2004) (see Exercise 15.1 on pp. 536-537) and is based on (Razin Sadka 1995).

1. The household's problem is to choose consumption, labor, capital and bonds holdings $\{c_t, n_t, k_{t+1}^H, b_{t+1}^H\}_{t=0}^{\infty}$, so as to solve

   $$\max_{\{c_t, n_t, k_{t+1}^H, b_{t+1}^H\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \tag{9}$$

   subject to the sequence of budget constraints:

   $$c_t + k_{t+1}^H + \frac{b_{t+1}^H}{R_t^*} \leq (1 - \tau_t^n) w_t n_t + r_t^* k_t^H + (1 - \delta) k_t^H + b_t^H \tag{10}$$

   for all $t = 0, 1, \ldots$, together with the transversality conditions:

   $$\lim_{T \to \infty} \left(\prod_{t=0}^{T-1} \frac{1}{R_t^*} \right) k_{T+1}^H = 0 \quad \text{and} \quad \lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{R_t^*} \right) b_{T+1}^H \tag{11}$$

2. No arbitrage imposes that

   $$R_t^* = r_{t+1}^* + (1 - \delta)$$

3. The government's time-t budget constraint is given by

   $$g_t + b_t^G \leq \hat{\tau}_t^k r_t^* k_t + \tau_t^n w_t n_t + \frac{b_{t+1}^G}{R_t^*} \tag{12}$$

   where b_t^G denotes government debt and k_t is the total capital stock (which may not necessarily be entirely owned by the households). Transversality conditions need also to be added to the government's budget constraint:

$$\lim_{T\to\infty}\left(\prod_{t=0}^{T-1}\frac{1}{R_t^*}\right)k_{T+1}=0 \quad \text{and} \quad \lim_{T\to\infty}\left(\prod_{t=0}^{T}\frac{1}{R_t^*}\right)b_{T+1}^G \tag{13}$$

4. Firms' profit maximization implies

$$F_k(k_t,n_t)=(1+\hat{\tau}_t^k)r_t^* \tag{14}$$

$$F_n(k_t,n_t)=w_t \tag{15}$$

5. At the optimum, the representative household's and the government's budget constraints would hold with equality for all t. Adding up (10) and (12), we get

$$c_t+k_{t+1}^H+\frac{b_{t+1}^H}{R_t^*}+g_t+b_t^G=(1-\tau_t^n)w_tn_t+r_t^*k_t^H+(1-\delta)k_t^H+b_t^H+\hat{\tau}_t^kr_t^*k_t+\tau_t^nw_tn_t+\frac{b_{t+1}^G}{R_t^*}$$

Using the fact that $F(k_t,n_t)$ is a CRS production function and thus homogeneous of degree one, we can make use of Euler's Theorem to write

$$(1+\hat{\tau}_t^k)r_t^*k_t+w_tn_t=F_k(k_t,n_t)k_t+F_n(k_t,n_t)n_t=F(k_t,n_t)$$

where we also used the FOC's (14) and (15). Substituting this into the combined budget constraint and rearranging terms, we get

$$c_t+k_{t+1}+g_t+(k_{t+1}^H-k_{t+1})+(b_t^G-b_t^H)=F(k_t,n_t)+\frac{b_{t+1}^G-b_{t+1}^H}{R_t^*}+(1-\delta+r_t^*)k_t^H-r_t^*k_t$$

6. Since both the government and the households can borrow and save in the international capital market, the resource constraint of the closed economy does not necessarily hold. First, b_t^G may be different from b_t^H, which means that all government bonds are not necessarily owned by domestic households. $b_t^G-b_t^H$ may be interpreted as the country deficit. Similarly k_t may be different from k_t^H . If $k_t>k_t^H$, then some domestic capital is owned by foreign investors. If on the other hand $k_t<k_t^H$, then domestic investors own foreign capital.

7. The FOC's of the household's problem would give

$$\beta^tu_c(c_t,1-n_t)=\lambda_0q_t^* \tag{16}$$

$$-\beta^tu_n(c_t,1-n_t)=\lambda_0q_t^*(1-\tau_t^n)w_t \tag{17}$$

where λ_0 is the Lagrange multiplier on the time zero budget constraint and

$$q_t^*=\prod_{j=0}^{t-1}\frac{1}{R_j^*}$$

8. The household's intertemporal budget constraint is given by

$$\sum_{t=0}^{\infty} q_t^* c_t \leq \sum_{t=0}^{\infty} q_t^* (1-\tau_t^n) w_t n_t + \left[(1-\delta) + r_0^*(1-\tau_0^k)\right] k_0^H + b_0 \tag{18}$$

Making use of (16) and (17) together with the normalization $q_0^* = 1$, (18) becomes

$$\sum_{t=0}^{\infty} \left[q_t^* c_t + \beta^t \frac{u_n(c_t, 1-n_t)}{u_c(c_0, 1-n_0)} n_t\right] \leq \left[1-\delta + r_0^*(1-\tau_0^k)\right] k_0^H + b_0 \tag{19}$$

The government's intertemporal budget constraint is given by

$$b_0^G + \sum_{t=0}^{\infty} q_t^* g_t \leq \sum_{t=0}^{\infty} q_t^* \left[\hat{\tau}_t^k r_t^* k_t + \tau_t^n w_t n_t\right] \tag{20}$$

Note that due to CRS, we can write the total tax collections as

$$\hat{\tau}_t^k r_t^* k_t + \tau_t^n w_t n_t = F(k_t, n_t) - r_t^* k_t - (1-\tau_t^n) w_t n_t$$

so that the intertemporal budget constraint of the government is

$$b_0^G + \sum_{t=0}^{\infty} q_t^* \left[g_t - F(k_t, n_t) + r_t^* k_t\right] - \frac{u_n(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} n_t \leq 0 \tag{21}$$

9. The primal approach to the Ramsey problem in this small economy context is to maximize the representative agent utility subject to the intertemporal budget constraint of the household (10) and the intertemporal budget constraint of the government (12). The Ramsey problem is to choose the sequence $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize the representative household's utility (9) subject to the intertemporal budget constraints (19) and (21). Observe that for $t \geq 1$, the capital stock k_t appears only in the government's budget constraint (21). Observe that the capital stock appears only in the government's budget constraint. Taking derivative in the Lagrangian with respect to k_t yields

$$F_k(k_t, n_t) = r_t^* \tag{22}$$

which, given (14), implies that $\hat{\tau}_t^k = 0$ for all $t \geq 1$.

10. As we can see, the result of (Chamley1986) and (Judd1985) naturally extends to this open-economy environment: optimality requires that there be no distortion in the capital taxation. Observe that for the open-economy version, we no longer need the assumption that utility is additively separable in consumption and labor and has intertemporal elasticity of substitution for consumption.[1]

[1] That assumption is needed for the closed-economy version, though: we must have $u(c,n) = \frac{1}{1-\sigma} c^{1-\sigma} - v(n)$ for some $\sigma \geq 0$ and increasing and convex $v(\cdot)$. Given the more general specification, the planner would generally wish to distort capital accumulation in order to affect state prices.

3 Two Labor Inputs

This problem is taken from (Ljungqvist Sargent2004) (see Exercise 15.4 on p. 538) and is based on (Jones et al1997).

1. The household's problem is to choose the sequences $\{c_t, n_{1t}, n_{2t}\}_{t=0}^{\infty}$ to solve

$$\max_{\{c_t, n_{1t}, n_{2t}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_{1t} - n_{2t}) \tag{23}$$

subject to the intertemporal budget constraint:

$$\sum_{t=0}^{\infty} q_t^0 (1+\tau_t^c) c_t \leq \sum_{t=0}^{\infty} q_t^0 \left[(1-\tau_t^{n_1}) w_{1t} n_{1t} + (1-\tau_t^{n_2}) w_{2t} n_{2t} \right] + \left[r_0 + (1-\delta) \right] k_0 + b_0 \tag{24}$$

The FOC's with respect to c_t, n_{1t} and n_{2t} are

$$c_t : \qquad \beta^t u_c(t) = \lambda q_t^0 (1+\tau_t^c) \tag{25}$$

$$n_{1t} : \qquad -\beta^t u_n(t) = \lambda q_t^0 (1-\tau_t^{n_1}) w_{1t} \tag{26}$$

$$n_{2t} : \qquad -\beta^t u_n(t) = \lambda q_t^0 (1-\tau_t^{n_2}) w_{2t} \tag{27}$$

Taking time-zero consumption as the numéraire (that is, setting $q_0^0 = 1$), we find

$$\beta^t \frac{u_c(t)}{u_c(0)} = q_t^0$$

and

$$-\beta^t \frac{u_l(t)}{u_c(0)} = q_t^0 (1-\tau_t^{n_1}) w_{1t} = q_t^0 (1-\tau_t^{n_2}) w_{2t}$$

Using these facts, let us rewrite the household's intertemporal budget constraint as

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t) c_t + u_n(t)(n_{1t} + n_{2t}) \right] = u_c(0) \left[(r_0 + (1-\delta)) k_0 + b_0 \right]$$

From now on, define $A \equiv u_c(0) \left[(r_0 + (1-\delta)) k_0 + b_0 \right]$. Define the value function by

$$V(c_t, n_{1t}, n_{2t}, \Phi) \equiv u(c_t, 1 - n_{1t} - n_{2t}) + \Phi (1+\tau_0^c) \left[u_c(t) c_t + u_n(t)(n_{1t} + n_{2t}) \right] \tag{28}$$

The Lagrangian associated with the Ramsey plan is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ V(c_t, n_{1t}, n_{2t}, \Phi) + \lambda_t \left[F(k_t, n_{1t}, n_{2t}) - (1-\delta) k_t - c_t - g_t - k_{t+1} \right] \right\} - \Phi A$$

Taking FOC's, we get

$$c_t: \quad V_c(t) = \lambda_t \tag{29}$$

$$n_{1t}: \quad -V_{n_1}(t) = \lambda_t F_{n_1}(t) \tag{30}$$

$$n_{2t}: \quad -V_{n_2}(t) = \lambda_t F_{n_2}(t) \tag{31}$$

$$k_{t+1}: \quad \lambda_t = \beta\lambda_{t+1}\left[F_k(t+1) + (1-\delta)\right] \tag{32}$$

$$c_0: \quad V_c(0) = \lambda_0 + \Phi A_c \tag{33}$$

$$n_{10}: \quad -V_{n_1}(0) = \lambda_0 F_{n_1}(0) - \Phi A_{n_1} \tag{34}$$

$$n_{20}: \quad -V_{n_2}(0) = \lambda_0 F_{n_2}(0) - \Phi A_{n_2} \tag{35}$$

where the first four equations have to be satisfied for all $t \geq 1$. Now, from the FOC's of the household's problem(26) and (27), we must have

$$(1-\tau_t^{n_1})w_{1t} = (1-\tau_t^{n_2})w_{2t}$$

for all $t = 0, 1, \ldots$ On the other hand, from the firm's problem we get

$$w_{1t} = F_{n_1}(t) \quad \text{and} \quad w_{2t} = F_{n_2}(t)$$

Since n_{1t} and n_{2t} enter $V(\cdot)$ *additively*,[2] we must have $V_{n_1}(t) = V_{n_2}(t)$, which from (30) and (31) implies that $F_{n_1}(t) = F_{n_2}(t)$ for all t, meaning that we must have

$$1-\tau_t^{n_1} = 1-\tau_t^{n_2} \quad \Longleftrightarrow \quad \tau_t^{n_1} = \tau_t^{n_2}$$

On the other hand, at $t = 0$ we do not have $F_{n_1}(0) = F_{n_2}(0)$, because time zero capital tax is fixed exogenously. The following applies instead:

$$\lambda_0\left[F_{n_1}(0) - F_{n_2}(0)\right] = \Phi\left[A_{n_1} - A_{n_2}\right] = \Phi u_c(0)(1-\tau_0^k)k_0\left[F_{kn_1}(0) - F_{kn_2}(0)\right]$$

where the last equality is found by differentiating A with respect to n_1 and n_2. Note that when $\Phi = 0$, i.e. when we don't put restrictions on τ_0^k , then we have $F_{n_1}(0) = F_{n_2}(0)$.

As before let us use the FOC's for the household's and the firm's problem to write that

$$(1-\tau_t^{n_1})F_{n_1}(0) = (1-\tau_t^{n_2})F_{n_2}(0)$$

[2]Indeed, $V_{n_{it}} = -u_n(t) + \Phi(1+\tau_0^c)\left[u_n(t) - u_{nn}(t)(n_{1t}+n_{2t})\right]$ for $i = 1, 2$. As can be seen, it depends on $n_{1t}+n_{2t}$, but not on n_{1t} and n_{2t} *separately*.

so that

$$\frac{F_{n_2}(0)}{F_{n_1}(0)} = \frac{1-\tau_0^{n_1}}{1-\tau_0^{n_2}}$$

Replacing this relationship in the equation above, we find

$$\lambda_0 F_{n_1}(0) \cdot \frac{\tau_0^{n_1}-\tau_0^{n_2}}{1-\tau_0^{n_2}} = \Phi u_c(0)(1-\tau_0^k)k_0\left[F_{kn_1}(0) - F_{kn_2}(0)\right]$$

Recall that λ_0 and Φ are Lagrange multipliers of inequality constraints and are therefore positive. We have: sign $\left[\tau_0^{n_1}-\tau_0^{n_2}\right]$ = sign $\left[F_{kn_1}(0) - F_{kn_2}(0)\right]$.

Now assume, for example, that k and n_1 are complements, so that the marginal return with respect to k is *increasing* in n_1. Also assume that k and n_2 are substitutes, so that the marginal return with respect to k is *decreasing* in n_2. This would imply

$$\tau_0^{n_1} > \tau_0^{n_2} \tag{36}$$

To gain some intuition for why this is the case, remember that the welfare cost of distortionary taxation is measured by the derivative of the Lagrangian with respect to τ_0^k:

$$\frac{\partial \mathcal{L}}{\partial \tau_0^k} = \Phi u_c(0) F_k(0) k_0$$

Note that if the marginal product of capital is high, then the welfare cost of using distortionary taxation is high – equivalently welfare could be improved a lot by decreasing τ_0^k. Hence, one can reduce the welfare cost of distortionary taxation by lowering the marginal product of capital. Therefore, the planner should choose a relatively high n_2 and a relatively low n_1, which is achieved by taxing n_2 less than n_1.

2. We follow the usual steps for the characterization of the Ramsey problem. The household's problem is now to choose $\{c_t, n_{1t}, n_{2t}\}_{t=0}^{\infty}$ to solve

$$\max_{\{c_t,n_{1t},n_{2t}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_{1t}, 1-n_{2t})$$

subject to

$$\sum_{t=0}^{\infty} q_t^0(1+\tau_t^c)c_t \leq \sum_{t=0}^{\infty} q_t^0\left[(1-\tau_t^{n_1})w_{1t}n_{1t} + (1-\tau_t^{n_2})w_{2t}n_{2t}\right] + \left[r_0 + (1-\delta)\right]k_0 + b_0$$

Using the restriction that for all t, $\tau_t^{n_1} = \tau_t^{n_2} = \tau_t^n$, the FOC's are

$$c_t: \quad \beta^t u_c(t) = \lambda q_t^0 (1+\tau_t^c)$$

$$n_{1t}: \quad -\beta^t u_{n_1}(t) = \lambda q_t^0 (1-\tau_t^n) w_{1t}$$

$$n_{2t}: \quad -\beta^t u_{n_2}(t) = \lambda q_t^0 (1-\tau_t^n) w_{2t}$$

As usual, taking time-0 consumption as the numéraise (that is, setting $q_0^0 = 1$), we find:

$$q_t^0 = \beta^t \frac{u_c(t)}{u_c(0)}$$

$$q_t^0(1-\tau_t^n)w_{1t} = -\beta^t \frac{u_{n_1}(t)}{u_c(0)}$$

$$q_t^0(1-\tau_t^n)w_{2t} = -\beta^t \frac{u_{n_2}(t)}{u_c(0)}$$

At this point, we can write the household's intertemporal budget constraint:

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t)c_t + u_{n_1}(t)n_{1t} + u_{n_2}(t)n_{2t}\right] = A$$

where $A \equiv u_c(0)\left[(r_0 + (1-\delta))k_0 + b_0\right]$, as before.

Now let us form the Lagrangian. We follow the hint and incorporate constraints forcing the planner to raise equal taxes on both labor inputs. Recall that from the FOC's of the household's problem, we have $(1-\tau_t^{n_i})w_{it} = -u_{n_i}(t)$, $i = 1, 2$. From the first order conditions of the firm's problem, we have $w_{it} = F_{n_i}(t)$, $i = 1, 2$. In a competitive equilibrium, both labor taxes are equal if and only if

$$-\frac{u_{n_1}(t)}{F_{n_1}(t)} = -\frac{u_{n_2}(t)}{F_{n_2}(t)} \quad \Longleftrightarrow \quad -u_{n_1}(t)F_{n_2}(t) = -u_{n_2}(t)F_{n_1}(t)$$

As before, define the value function as

$$V(c_t, n_{1t}, n_{2t}, \Phi) \equiv u(c_t, 1-n_{1t}, 1-n_{2t}) + (1-\tau_0^c)\Phi\left[u_c(t)c_t + u_{n_1}(t)n_{1t} + u_{n_2}(t)n_{2t}\right]$$

The associated Lagrangian is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{V(c_t, n_{1t}, n_{2t}, \Phi) + \lambda_t \left[F(k_t, n_{1t}, n_{2t}) - (1-\delta)k_t - c_t - g_t - k_{t+1}\right]\right\}$$

$$+ \sum_{t=0}^{\infty} \beta^t \mu_t \left\{u_{n_2}(t)F_{n_1}(t) - u_{n_1}(t)F_{n_2}(t)\right\} - \Phi A$$

The FOC's are given by

$$c_t: \qquad V_c(t) = \lambda_t - \mu_t \left[u_{n_2c}(t)F_{n_1}(t) - u_{n_1c}(t)F_{n_2}(t)\right]$$

$$\begin{aligned} n_{1t}: \qquad -V_{n_1}(t) &= \lambda_t F_{n_1}(t) \\ &\quad - \mu_t \left[u_{n_1n_1}(t)F_{n_2}(t) + u_{n_1}(t)F_{n_1n_2}(t) - u_{n_1n_2}(t)F_{n_1}(t) - u_{n_1}(t)F_{n_1n_1}(t)\right] \end{aligned}$$

$$\begin{aligned} n_{2t}: \qquad -V_{n_2}(t) &= \lambda_t F_{n_2}(t) \\ &\quad - \mu_t \left[u_{n_1n_2}(t)F_{n_2}(t) + u_{n_1}(t)F_{n_2n_2}(t) - u_{n_2n_2}(t)F_{n_1}(t) - u_{n_1}(t)F_{n_1n_2}(t)\right] \end{aligned}$$

$$\begin{aligned} k_{t+1}: \qquad \lambda_t &= \beta\lambda_{t+1}\left[F_k(t+1) + (1-\delta)\right] \\ &\quad - \beta\mu_{t+1}\left[u_{n_2}(t+1)F_{n_1k}(t+1) - u_{n_1}(t+1)F_{n_2k}(t+1)\right] \end{aligned}$$

$$c_0: \qquad V_c(0) = \lambda_0 - \mu_0 \left[u_{n_2c}(0)F_{n_1}(0) - u_{n_1c}(0)F_{n_2}(0)\right] + \Phi A_c$$

$$\begin{aligned} n_{10}: \qquad -V_{n_1}(0) &= \lambda_0 F_{n_1}(0) \\ &\quad - \mu_0 \left[u_{n_1n_1}(0)F_{n_2}(0) + u_{n_1}(0)F_{n_1n_2}(0) - u_{n_1n_2}(0)F_{n_1}(0) - u_{n_1}(0)F_{n_1n_1}(0)\right] \\ &\quad - \Phi A_{n_1} \end{aligned}$$

$$\begin{aligned} n_{20}: \qquad -V_{n_2}(0) &= \lambda_0 F_{n_2}(0) \\ &\quad - \mu_0 \left[u_{n_1n_2}(0)F_{n_2}(0) + u_{n_1}(0)F_{n_2n_2}(0) - u_{n_2n_2}(0)F_{n_1}(0) - u_{n_1}(0)F_{n_1n_2}(0)\right] \\ &\quad - \Phi A_{n_1} \end{aligned}$$

with the first four conditions valid for $t \geq 1$. Together with

$$c_t + g_t + k_{t+1} = F(k_t, n_{1t}, n_{2t}) + (1-\delta)k_t$$

$$\sum_{t=0}^{\infty} \beta^t \left[u_c(t)c_t + u_{n_1}(t)n_{1t} + u_{n_2}(t)n_{2t}\right] - A = 0$$

$$\mu_t \left\{u_{n_2}(t)F_{n_1}(t) - u_{n_1}(t)F_{n_2}(t)\right\} = 0$$

they form the system of differential equations that characterizes the solution to the Ramsey planning problem.

3. Assume that the solution to this Ramsey converges to a steady state for which the constraint that the two labor taxes should be equal binds. Thus $\mu_t \to \mu \neq 0$ as $t \to \infty$. The steady-state version of the FOC associated with k_{t+1} can be written as

$$1 = \beta \left\{F_k + (1-\delta) + \frac{\mu}{\lambda}\left[u_{n_2}F_{n_1k} - u_{n_1}F_{n_2k}\right]\right\} \tag{37}$$

On the other hand, the no-arbitrage condition for capital is, in steady state:

$$\frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}\frac{u_c(t)}{u_c(t+1)} = \frac{1}{\beta} = (1-\tau_{t+1}^k)F_k + (1-\delta) \tag{38}$$

where the second equality makes use of the fact that the steady-state consumption is constant. Combining these last two equations yields

$$\tau_{t+1}^k = -\frac{\mu}{\lambda.F_k}\left\{u_{n_2}F_{n_1k} - u_{n_1}F_{n_2k}\right\} \tag{39}$$

which is different from zero unless $u_{n_2}F_{n_1k} - u_{n_1}F_{n_2k} = 0$. Finally, recall that

$$u_{n_2}F_{n_1k} = u_{n_1}F_{n_2k} \quad \Longleftrightarrow \quad (1-\tau^n)w_1F_{n_2k} = (1-\tau^n)w_2F_{n_1k} \tag{40}$$

which, given that $w_1 = F_{n_1}$ and $w_2 = F_{n_2}$, implies

$$F_{n_1}F_{n_2k} = F_{n_2}F_{n_1k} \tag{41}$$

We conclude that the steady state tax on capital is *not* zero unless $F_{n_1}F_{n_2k} = F_{n_2}F_{n_1k}$.

4 Ramsey Taxation

This exercise considers a simplified version of the Ramsey problem with complete markets seen in class. The latter parts of the problem then modify the asset market structure to conjecture what happens in incomplete market environments. The two simplifications are: (i) no capital accumulation, only labor taxes and (ii) a specific process for uncertainty with a single shock at date 1, and no further uncertainty afterwards. We saw in the generalized Ramsey framework that with complete markets labor tax distortions should be smoothed across time, and debt/assets be used to shift resources over time and across states. With power utilities, this implied constant labor tax rates.

The subsequent asset market structures alter the ability to shift resources over time. With one period bonds, there is limited ability to shift resources across states in period 1, but perfect smoothing across time, so the shock should lead to a fluctuation in labor taxes in period 1, but not in future periods. With one period bonds and perpetuities, two securities are again sufficient for perfect smoothing across time and states: after date 1, the second security is redundant. At date 1, the gap in their one period returns, will provide full spanning of the spending risk, and hence recover the full tax smoothing result.

1. The simplest way to analyze the problem is by working backwards. Let $V(b)$ denote the value of the deterministic Ramsey problem if there is an initial debt level b, and no further government spending obligations – this is the problem starting from period 2 after a disaster, and starting from period 1, if there is no disaster.

   $V(b)$ is defined as follows. In any period, for a given labor tax τ , the resulting labor supply solves the first-order condition $(1-\tau)u'(n) = v'(n)$, where we have used the condition that $n = y = c$. Let $\tau(n) = 1 - v'(n)/u'(n)$ denote the tax rate that generates a labor supply n, and let $w(n) = u(n) - v(n)$ denote the resulting per period utility, in terms of n. The problem is then defined as

$$V(b) = \max_{\{n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t w(n_t) \tag{42}$$

subject to

$$b \leq \sum_{t=0}^{\infty} \beta^t \frac{u'(n_t)}{u'(n_0)} \tau(n_t) n_t \tag{43}$$

It is immediate that the solution to this problem is stationary over time, and prescribes a constant tax rate τ, such that $b = \tau(n)n/(1-\beta)$. Writing the resulting value function in terms of n, we have $V(b(n)) = w(n)/(1-\beta)$.

Likewise, the period 0 utility is $w(n_0)$. With this, we can write the planner objective in terms of the labor supply in all periods as:

$$w(n_0) + \frac{\beta}{1-\beta}(1-\pi)w(n_N) + \frac{\beta^2}{1-\beta}\pi w(n_D) + \beta\pi\left(u(\hat{n}_D - G) - v(\hat{n}_D)\right)$$

where n_0 is the labor supply in period 0, n_N is labor supply in periods $1, 2, \ldots$ if no disaster occurs, $\hat{n}_D$ is labor supply in period 1 if a disaster occurs, and n_D is labor supply in periods $2, 3, \ldots$ after a disaster has occurred.

In addition, we need to keep track of the first-order condition in period 1 in the disaster state:

$$(1 - \hat{\tau}_D)u'(\hat{n}_D - G) = v'(\hat{n}_D) \tag{44}$$

The government revenue at any history is τn, which has to be weighted by the state-prices (discounted probability-weighted marginal utilities). This leads to the following government budget constraint:

$$\tau(n_0)n_0u'(n_0) + \frac{\beta}{1-\beta}(1-\pi)\tau(n_N)n_Nu'(n_N) + \frac{\beta^2}{1-\beta}\pi\tau(n_D)n_Du'(n_D)$$

$$+ \beta\pi\left((\hat{n}_D - G)u'(\hat{n}_D - G) - \hat{n}_D v'(\hat{n}_D)\right) \geq 0 \tag{45}$$

We have thus formulated the Ramsey planner objective and implementability constraint just in terms of the labor allocations.

2. Now, from the stationarity of the problem above, it is immediate that at the optimum, $n_0 = n_N = n_D$, i.e. the labor supply and tax rate should be the same whenever $G = 0$. Working out the first-order condition for $\hat{n}_D$ and n_0, we find the same tax-smoothing result as in class, i.e. that the tax distortions in the disaster and no-disaster states are also the same, if preferences admit a power function representation. Notice that it implies that the labor supply in the disaster state $\hat{n}_D$ is higher than in the other states in this case:

$$\hat{\tau_D} = \tau_0 \Rightarrow \frac{v'(\hat{n_D})}{u'(\hat{n_D} - G)} = \frac{v'(n_0)}{u'(n_0)}$$

3. With uncontingent bonds, perfect smoothing between the disaster and no-disaster histories is no longer possible. In particular, if the government saves a in uncontingent bonds at date 0, then it needs to raise a revenue of $G - a$ after the disaster state, while in the no-disaster state it is zero – the interest payments on the bonds are simply rebated back to the households. The resulting plannerï£¡s value function takes the same form as above, but adds as a restriction that $n_N = n_N^*$ (the first-best labor supply), and $\tau_N = 0$ (no revenue raised from no-disaster states). As a result, the no-disaster states can no longer be used to reduce the distortions needed to finance the spending in the disaster state.

4. With uncontingent bonds and perpetuities, we show that perfect smoothing again becomes possible. Now, for all periods from $t > 1$ onwards, there is no further uncertainty, and the two assets are redundant, implying that the perpetuity will be priced at $\beta/(1-\beta)$ in all future periods (ex dividend). At date 1, the state-contingent return for a perpetuity is

   $$\begin{cases} \frac{1}{1-\beta}, & \text{if there is no disaster} \\ 1 + \frac{u'(\hat{n}_D)}{u'(\hat{n}_D - G)} \frac{1}{1-\beta}, & \text{if there is a disaster} \end{cases}$$

   while the return from the bond is uncontingent.

   Therefore, as long as $\frac{u'(\hat{n}_D)}{u'(\hat{n}_D - G)} \neq 1$, the government can exploit the fluctuation in the value of the perpetuity after a disaster, along with the uncontingent bonds, in order to fully span the state space, and therefore perfectly hedge against the risk of government spending. As a result, complete contingent markets are restored and the government can implement the complete markets solution from (a).