Macroeconomics II – Problem Set 3

Solutions

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1 Jump Process

a. Recall that a discrete random variable n distributed according to a Poisson distribution with parameter λt has *p.d.f*

$$P(n = k|\lambda t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

where the parameter λt is the expected number of occurrences in a given time period. Then, the probability of $n = 0$ is,

$$P(n = 0|\lambda t) = e^{-\lambda t}$$

which imply,

$$P(\text{the event occurs at } t \leq \tau) = P(n \geq 1|\lambda t) = 1 - P(n = 0|\lambda t) = 1 - e^{-\lambda t}$$

b.

$$\begin{aligned} P(t \leq \tau | t > s) &= \frac{P(t \leq \tau, t > s)}{P(t > s)} \\ &= \frac{P(t \leq \tau) - P(t \leq s)}{1 - P(t \leq s)} \\ &= \frac{(1 - e^{-\lambda\tau}) - (1 - e^{-\lambda s})}{1 - (1 - e^{-\lambda s})} \\ &= \frac{-e^{-\lambda\tau} + e^{-\lambda s}}{e^{-\lambda s}} \\ &= 1 - e^{-\lambda(\tau - s)} \\ &= P(t \leq \tau - s) \end{aligned}$$

c.

$$
\begin{aligned}
P(t \le \tau | t > s) &= \frac{P(t \le \tau, t > s)}{P(t > s)} \\
&= \frac{P(t \le \tau) - P(t \le s)}{1 - P(t \le s)} \\
&= \frac{(1 - e^{-\int_0^\tau \lambda_k dk}) - (1 - e^{-\int_0^s \lambda_k dk}))}{1 - (1 - e^{-\int_0^s \lambda_k dk})} \\
&= \frac{-e^{-\int_0^\tau \lambda_k dk} - e^{-\int_0^s \lambda_k dk}}{e^{-\int_0^s \lambda_k dk}} \\
&= 1 - e^{-\int_0^\tau \lambda_k dk - \int_0^s \lambda_k dk} \\
&= 1 - e^{-\int_s^\tau \lambda_k dk}
\end{aligned}
$$

$$P(t \le \tau - s) = 1 - e^{-\int_0^{\tau - s} \lambda_k dk}$$

If λ_t is not constant, $P(t \le \tau | t > s) \ne P(t \le \tau - s)$ so this process is not memoryless.

If λ_t is constant,

$$P(t \le \tau | t > s) = 1 - e^{-\int_s^\tau \lambda dk} = 1 - e^{-\lambda(\tau - s)} = 1 - e^{-\int_0^{\tau - s} \lambda dk}$$

$P(t \le \tau | t > s) = P(t \le \tau - s)$ so this process is memoryless.

d. Denote W_t as the value of being employed starting at time t, and U_t as the value of being unemployed starting at time t.

The value of being unemployed at time t equals the expected value of getting a job at some future time $s \in [t, +\infty)$

$$U_t = \int_t^\infty f(\text{a job arrives at s}) \text{ the value of a job of s}$$

We assume a Poisson Process, so the density of a job arriving at time s is $\frac{dP(\tau \le | \tau > t)}{ds} = \lambda_s e^{-\int_t^s \lambda_k dk}$.

When a job offer arrives at data s, the worker can take and become employed or decline it and remain unemployed. Thus the value of an offer of time s at time t is

$$\int_t^s e^{-r(\tau - t)} b_\tau d\tau + e^{-r(s-t)} max\{W_\tau, U_\tau\}$$

In our case, the job pays w for ever and we assume $b_t < w$, so $W_t \ge U_t$, $\forall t$. So,

$$max\{W_\tau, U_\tau\} = W_t, \forall t$$

Moreover,

$$
\begin{aligned}
W_t &= \int_t^\infty e^{-r(\tau - t)} w d\tau \\
&= w \frac{1}{r} (-e^{-r(\tau - t)})|_t^\infty \\
&= \frac{w}{r}
\end{aligned}
$$

So the value of being unemployed , which equals the expected value of getting a job, is

$$
\begin{aligned}
U_t &= \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} [\int_t^s e^{-r(\tau-t)} b_\tau d\tau + e^{-r(s-t)} max\{W_\tau, U_\tau\}] ds \\
&= \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} [\int_t^s e^{-r(\tau-t)} b_\tau d\tau + e^{-r(s-t)} W_\tau] ds \\
&= \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} [\int_t^s e^{-r(\tau-t)} b_\tau d\tau + e^{-r(s-t)} \frac{w}{r}] ds
\end{aligned}
$$

Differentiate both sides of the above equation wrt time (t)

$$
\begin{aligned}
\dot{U}_t &= \lambda_t e^{-\int_t^t \lambda_k dk} [\int_t^t e^{-r(\tau-t)} b_\tau d\tau + e^{-r(t-t)} \frac{w}{r}] \\
&+ \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} [\int_t^s \frac{\partial}{\partial t} e^{-r(\tau-t)} b_\tau d\tau + e^{-r(s-t)} \frac{w}{r}] ds \\
&= -\lambda_t \frac{w}{r} + \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} \lambda_t [\int_t^s e^{-r(\tau-t)} b_\tau d\tau + e^{-r(s-t)} \frac{w}{r}] ds \\
&+ \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} [\int_t^s e^{-r(t-t)} b_t + e^{-r(s-t)} \frac{w}{r}] ds \\
&= -\lambda_t \frac{w}{r} + \lambda_t U_t - b_t \int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} ds + rU_t \\
&= -\lambda_t \frac{w}{r} + \lambda_t U_t - b_t + rU_t
\end{aligned}
$$

where we use,

$$\frac{\partial}{\partial s} [-e^{-\int_t^s \lambda_k dk}] = \lambda_s e^{-\int_t^s \lambda_k dk}$$

to get,

$$\int_t^{\infty} \lambda_s e^{-\int_t^s \lambda_k dk} ds = -e^{-\int_t^s \lambda_k dk} |_t^{\infty} = 0 + 1 = 1$$

So we get the Bellman equations

$$W_t = \lambda_t \frac{w}{r}$$

$$rU_t = \dot{U}_t + b_t + \lambda_t (\frac{w}{r} - U_t)$$

We can interpret the last equation as equating the opportunity cost of unemployment in the LHS, with the expected gains from being unemployed in the RHS, containing (1) the capital gains from changes in the value of unemployment; (2) flow of unemployment benefit; (3) expected gains from becoming employed.

2 Unemployment insurance

a. An increase in τ reduces the wage w. We show this in the following steps. Our object is to find the wage w as a function of parameters of the model only.

   Sept 1: Write the Bellman Equations for U,W,J,V in steady state.

In this set-up, 1) the output of a filled job is $p-\tau$; 2) We assume a Poisson Process with a parameter λ, so a job arrives at a probability of λ; 3) We assume the matching function is of the form

$$m(u,v)=\sqrt{uv}$$

So the Poisson rate at which vacancies find workers is

$$q(\theta)=q(\frac{v}{u})=\frac{m(u,v)}{v}=\sqrt{\frac{u}{v}}=\frac{1}{\sqrt{\theta}}$$

and the rate at which an unemployed worker finds a job is

$$q(\theta)\theta=\frac{v}{u}\sqrt{\frac{u}{v}}=\sqrt{\theta}$$

In steady state, capital gains from the change in the value of each state are zero, i.e., $\dot{U}=\dot{W}=\dot{J}=\dot{V}=0$. By the same logic in class, we have,

$$rV=-pc+\frac{1}{\sqrt{\theta}}(J-V) \tag{2.1}$$

$$rJ=p-\tau-w+\lambda(V-J) \tag{2.2}$$

$$rW=w+\lambda(U-W) \tag{2.3}$$

$$rU=z+\sqrt{\theta}(W-U) \tag{2.4}$$

Sept 2: Find the Nash bargaining solution for the wage.

We assume a Nash bargaining solution for the wage. That is the worker earning a share $\beta\in(0,1)$ for the match surplus. We have that the wage solves,

$$\max_{w}(W-U)^{\beta}(J-V)^{1-\beta} \tag{2.5}$$

s.t. (2.1)-(2.4)

w only appears in (2.2) and (2.3), from (2.2) and (2.3), we have

$$J=\frac{p-\tau-w+\lambda V}{r+\lambda} \qquad J=\frac{w+\lambda U}{r+\lambda} \tag{2.6}$$

Take F.O.C. wrt w of (2.5),

$$\beta(W-U)^{\beta-1}(J-V)^{1-\beta}\frac{\partial W}{\partial w}+(1-\beta)(W-U)^{\beta}(J-V)^{-\beta}\frac{\partial J}{\partial w}=0$$

$$\beta(W-U)^{\beta-1}(J-V)^{1-\beta}\frac{1}{r+\lambda}+(1-\beta)(W-U)^{\beta}(J-V)^{-\beta}\frac{-1}{r+\lambda}=0$$

$$\beta(J-V)=(1-\beta)(W-U) \tag{2.7}$$

From (2.1) and (2.2), we have,

$$r(J-V)=p(1+c)-\tau-w-(\lambda+\frac{1}{\sqrt{\theta}})(J-V)$$

$$\beta r(J-V)=\beta[p(1+c)-\tau-w]-\beta(\lambda+\frac{1}{\sqrt{\theta}})(J-V)$$

From (2.3) and (2.4), we have,

$$r(W-U) = w - z - (\sqrt{\theta}+\lambda)(W-U)$$

$$(1-\beta)r(W-U) = (1-\beta)(w-z) - (1-\beta)(\sqrt{\theta}+\lambda)(W-U)$$

Using (2.7),

$$\beta[p(1+c)-\tau-w] - \beta(\lambda+\frac{1}{\sqrt{\theta}})(J-V) = (1-\beta)(w-z) - (1-\beta)(\sqrt{\theta}+\lambda)(W-U)$$

Using (2.7) again,

$$\beta[p(1+c)-\tau-w] - \beta(\lambda+\frac{1}{\sqrt{\theta}})(J-V) = (1-\beta)(w-z) - \beta(\sqrt{\theta}+\lambda)(J-V)$$

$$0 = (w-z) - \beta[p(1+c)-\tau-z] - \beta(\sqrt{\theta}-\frac{1}{\sqrt{\theta}})(J-V)$$

$$w = (1-\beta)z + \beta[p(1+c)-\tau] - \beta(\frac{1}{\sqrt{\theta}+\sqrt{\theta}})(J-V) \tag{2.8}$$

Step 3: Use the free entry condition to get rid of $J-V$.

Free entry (of posting a vacancy) implies $V=0$, using (2.1), we have,

$$0 = -pc + \frac{1}{\sqrt{\theta}}J$$

$$J = pc\sqrt{\theta} \tag{2.9}$$

Combining with (2.8), we have,

$$w = (1-\beta)z + \beta[p(1+c)-\tau] + \beta(\frac{1}{\sqrt{\theta}-\sqrt{\theta}})J$$

$$w = (1-\beta)z + \beta[p(1+c)-\tau] + \beta(\frac{1}{\sqrt{\theta}-\sqrt{\theta}})pc\sqrt{\theta}$$

$$w = (1-\beta)z + \beta(p-\tau) + \beta pc\theta \tag{2.10}$$

So $\frac{\partial w}{\partial \tau} = -\beta < 0$. An increase in τ reduces the wage w. The intuition is that the tax reduces the firm's profit, thus the surplus of a match and by rent-sharing some (a fraction β) of this loss is reflected in wages.

b. In steady state, we have the Beveridge Curve, i.e., flows into unemployment equals flows out of unemployment,

$$\sqrt{\theta}u = \lambda(1-u)$$

$$u = \frac{\lambda}{\lambda+\sqrt{\theta}} \tag{2.11}$$

Government balances the budget implies,

$$uz = (1-u)\tau$$

$$\tau = \frac{u}{1-u}z = \frac{\frac{\lambda}{\lambda+\sqrt{\theta}}}{1-\frac{\lambda}{\lambda+\sqrt{\theta}}}z = \frac{\lambda}{\sqrt{\theta}}z \tag{2.12}$$

Pledging into (2.10),

$$w = (1-\beta)z + \beta(p - \frac{\lambda}{\sqrt{\theta}}z) + \beta pc\theta \tag{2.13}$$

So (2.12) and (2.13) offers the equation of τ and w in terms of θ, for a given benefit z.

The extra positive effect of θ on wages comes from the fact that the higher θ, the lower unemployment, the fewer people have to be paid the unemployment insurance benefit z and the more people will share the tax burden, so the per capita tax is lower for both reasons and, correspondingly, the wage is higher.

c. Free entry implies $V = 0$, combining (2.6) and (2.9)

$$J = \frac{p-\tau-w}{r+\lambda} \qquad J = pc\sqrt{\theta}$$

we have,

$$(r+\lambda)pc\sqrt{\theta} = p - \tau - w$$

Combining with (2.12) and (2.13),

$$(r+\lambda)pc\sqrt{\theta} = p - \frac{\lambda}{\sqrt{\theta}}z - (1-\beta)z - \beta(p - \frac{\lambda}{\sqrt{\theta}}z) - \beta pc\theta$$

$$(r+\lambda)pc\sqrt{\theta} = (1-\beta)p - (1-\beta)(1 + \frac{\lambda}{\sqrt{\theta}})z - \beta pc\theta$$

$$(r+\lambda)c\sqrt{\theta} = (1-\beta) - (1-\beta)(1 + \frac{\lambda}{\sqrt{\theta}})\frac{z}{p} - \beta c\theta$$

$$(r+\lambda)c\sqrt{\theta^*} + \beta c\theta^* = (1-\beta)(1 - \frac{\lambda + \sqrt{\theta^*}}{\sqrt{\theta^*}}\frac{z}{p}) \tag{2.14}$$

(2.14) implicitly defines θ^* as a function of $\frac{z}{p}$.

The Beveridge Curve(2.11) and the definition of θ gives equilibrium levels of unemployment u and vacancies v.

$$u^* = \frac{\lambda}{\lambda + \sqrt{\theta^*}}$$
$$v^* = u^*\theta^*$$

d. Solve for $\frac{z}{p}$ as a function of θ^* and study this inverse function is much easier. From (2.14),

$$(r+\lambda)c\sqrt{\theta^*} + \beta c\theta^* = (1-\beta)(1 - \frac{\lambda + \sqrt{\theta^*}}{\sqrt{\theta^*}}\frac{z}{p})$$

$$(1-\beta)\frac{\lambda + \sqrt{\theta^*}}{\sqrt{\theta^*}}\frac{z}{p} = (1-\beta) - (r+\lambda)c\sqrt{\theta^*} - \beta c\theta^*$$

$$\frac{z}{p} = \frac{(1-\beta) - (r+\lambda)c\sqrt{\theta^*} - \beta c\theta^*}{(1-\beta)\frac{\lambda+\sqrt{\theta^*}}{\sqrt{\theta^*}}}$$

$$\frac{z}{p} = \frac{(1-\beta)\sqrt{\theta^*} - (r+\lambda)c\theta^* - \beta c\theta^*\sqrt{\theta^*}}{(1-\beta)(\lambda + \sqrt{\theta^*})}$$

When θ^* is small,

$$\frac{z}{p} \approx \frac{(1-\beta)\sqrt{\theta^*}}{(1-\beta)(\lambda+\sqrt{\theta^*})} \qquad \text{is increasing in} \quad \theta^*$$

When θ^* is large,

$$\frac{z}{p} \approx \frac{-\beta c\theta^*\sqrt{\theta^*}}{(1-\beta)(\lambda+\sqrt{\theta^*})} \qquad \text{is decreasing in} \quad \theta^*$$

Thus, θ^* is decreasing in $\frac{z}{p}$ when θ^* is not small.

e. From (2.14), clearly the equilibrium is at most unique. BUT, when we write the free entry condition, we also need to ask if, at the equilibrium θ^*, posting an extra vacancy may in fact increase equilibrium profits. Usually (in the standard model) it does not, as an extra vacancy both reduces the vacancy-filling rate $q(\theta)$ and increases the workers outside option and wage, thus reducing J. As a consequence, the expected profits from posting a vacancy $q(\theta)J - pc$ fall and, if they are zero at the free entry condition, they becoming negative as more vacancies are posted. But in this problem an extra vacancy has another effect: it increases employment and it reduces unemployment, thus it reduces the tax for both reasons, and this may actually raise profits and offset the other effects, so an extra vacancy may make zero profits actually positive. In this case, assuming that firms internalize this effect, there would be no zero profits equilibrium, as an infinite number of firms would enter until all workers are employed. In other words, the only equilibrium would be one of full employment. This can be ruled out by the following reasoning. If there was zero unemployment, so zero tax, still the returns to posting a vacancy J would be finite, so given the cost pc the vacancy rate θ^* would be finite as in the standard texbook model. With finite θ^* unemployment must be positive, a contradiction.