Problem Set 4: Endogenous Growth Model (Solutions)

TAs: Yue FEI and Lan LAN
Previous TA: Alban Moura

1 A semi-endogenous growth model

1. Let W_t denote the net wealth of the representative household at date t, with the usual law of motion

$$\frac{dW_t}{dt} = r_t W_t + w_t L + D_t - C_t,$$

where r_t is the real interest rate in bonds, w_t is the real wage, D_t are dividends pain by firms, and C_t is aggregate consumption. Imposing the usual solvency condition

$$\lim_{t\to\infty} \exp\left(-\int_0^t r_s d_s \right) W_t \geq 0,$$

the household's intertemporal budget constraint writes

$$W_0 + \int_0^\infty \exp\left(-\int_0^t r_s ds \right)(w_t L + D_t - C_t)dt \geq 0. \tag{1}$$

More details in TD section.

2. The intertemporal allocation of aggregate consumption is obtained by solving the following static problem

$$\max_{C_t} \quad \int_0^\infty e^{-\rho t} \ln\left(\frac{C_t}{L_t}\right) dt$$
$$\text{s.t.} \quad W_0 + \int_0^\infty \exp\left(-\int_0^t r_s ds \right)(w_t L_t + D_t - C_t)dt \geq 0.$$

Taking the first-order condition with respect to C_t, and letting λ denote the Lagrange multiplier, we have

$$\frac{e^{-\rho t}}{C_t} = \lambda \exp\left(-\int_0^t r_s ds \right) \qquad \Longleftrightarrow \qquad C_t = \frac{e^{-\rho t}}{\lambda} \exp\left(\int_0^t r_s ds \right).$$

Differentiating with respect to time yields the usual Euler equation for consumption

$$\frac{dC_t/dt}{C_t} = r_t - \rho.$$

Moreover, using that $c_t := C_t/L_t$ and that $dL_t/dt = nL_t$, this equation is equivalent to

$$\frac{dc_t/dt}{c_t} = r_t - n - \rho. \tag{2}$$

3. In this Dixit-Stiglitz world, the aggregate ideal price index p_t is defined as

$$P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}.$$

We will discuss below why this is the right definition for the aggregate price index of consumption in this economy.

4. First, we compute the optimal consumer demand for each variety i, given the p_{it}'s. The consumer's problem writes

$$\max_{C_{it}} \quad \int_0^{N_t} C_{it}^{\frac{\sigma-1}{\sigma}} di$$
$$\text{s.t.} \quad \int_0^{N_t} p_{it} C_{it} di \leq E_t,$$

where C_{it} is total (and not per-capita) consumption of good i and E_t denotes consumption expenditures at date t. The first-order condition for a maximum is

$$\frac{\sigma-1}{\sigma} C_{it}^{-\frac{1}{\sigma}} = \lambda p_{it} \qquad \Longleftrightarrow \qquad C_{it} = \mu p_{it}^{-\sigma} \text{ with } \mu := \left(\frac{\sigma-1}{\sigma\lambda}\right)^{\sigma}. \tag{3}$$

Substituting C_{it} by (3) in the budget constraint, that is binding at the optimum, yields

$$\int_0^{N_t} \mu p_{it}^{1-\sigma} di = E_t \qquad \Longleftrightarrow \qquad \mu = \frac{E_t}{\int_0^{N_t} p_{it}^{1-\sigma} di}.$$

It then follows from (3) that the optimal demand function for variety i writes

$$C_{it} = \frac{E_t}{\int_0^{N_t} p_{it}^{1-\sigma} di} p_{it}^{-\sigma} = \frac{E_t}{P_t^{1-\sigma}} p_{it}^{-\sigma} = \frac{E_t}{P_t} \left(\frac{p_{it}}{P_t}\right)^{-\sigma}.$$

To get an intuition on the ideal price index, plug in these optimal demand functions in the definition of aggregate consumption to get

$$C_t = \left(\int_0^{N_t} C_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} = E_t P_t^{\sigma-1} \left(\int_0^{N_t} p_{it}^{1-\sigma} di \right)^{\frac{\sigma}{\sigma-1}} = E_t P_t^{\sigma-1} P_t^{-\sigma} = \frac{E_t}{P_t}.$$

Therefore, $P_t C_t = E_t$ and P_t can be seen as the price of the aggregate consumption good C_t.

Given the consumer's optimal demand functions, firm i's problem at date t writes

$$\max_{p_{it}} \ \pi_{it} = p_{it} C_{it}(p_{it}) - w_t C_{it}(p_{it}) = Y_t P_t^{\sigma-1} p_{it}^{1-\sigma} - w_t Y_t P_t^{\sigma-1} p_{it}^{-\sigma},$$

where we have used the fact that $l_{it} = C_{it}$. The optimal price must then verify the following condition

$$(\sigma-1)p_{it}^{-\sigma}=\sigma w_t p_{it}^{-\sigma-1} \qquad \Longleftrightarrow \qquad p_{it}=\frac{\sigma}{\sigma-1}w_t. \tag{4}$$

Hence, the optimal pricing policy is to charge a constant markup $\sigma/(\sigma-1)>1$ over the marginal cost, which is just the real wage rate here. A direct implication of this result is that all firms will charge the same price and that all varieties will be produced in the same quantity in equilibrium.

5. Taking the ideal price index P_t as the numeraire, it follows from (4) that

$$\left(\int_0^{N_t} p_{it}^{1-\sigma}di\right)^{\frac{1}{1-\sigma}}=1 \quad \Longrightarrow \quad \frac{\sigma}{\sigma-1}w_t N_t^{\frac{1}{1-\sigma}}=1 \quad \Longleftrightarrow \quad w_t=\frac{(\sigma-1)N_t^{\frac{1}{\sigma-1}}}{\sigma}, \tag{5}$$

an equation that expresses the real wage as a function of N_t.

6. Denote the fraction of the labor force employed in R&D at date t by β_t, so that there are $(1-\beta_t)L_t$ workers employed in the production sector. Since all goods are produced in the same quantity in equilibrium and since the production function for variety i is just $C_i=l_i$, the quantity of any variety i produced in equilibrium is

$$C_{it}=\frac{(1-\beta_t)L_t}{N_t}, \tag{6}$$

or

$$c_{it}=\frac{1-\beta_t}{N_t} \tag{7}$$

in per-capita terms.

7. Using the same symmetry argument, all monopolists will have the same equilibrium profits, given by

$$\begin{aligned}\pi_t=(p_{it}-w_t)C_{it}&=\left(\frac{\sigma}{\sigma-1}-1\right)\times\frac{(\sigma-1)N_t^{\frac{1}{\sigma-1}}}{\sigma}\times\frac{(1-\beta_t)L_t}{N_t}\\&=\frac{1}{\sigma}N_t^{\frac{2-\sigma}{\sigma-1}}(1-\beta_t)L_t,\end{aligned}$$

where the first line uses the expressions for the optimal price (4), the equilibrium wage (5), and the equilibrium production of all varieties (6).

8. As discussed in the previous exercise, the value of a firm V_t follows the Bellman equation

$$r_t V_t=\pi_t+\frac{dV_t}{dt}. \tag{8}$$

9. Free-entry in the R&D sector implies that innovators should make zero profit (cost of innovation equals benefit of innovation) in equilibrium, so that

$$\frac{w_t}{\gamma N_t^{\delta}}=V_t. \tag{9}$$

where cost of 1 innovation is $\frac{L_{Rt}}{\dot{N}_t} = \frac{1}{\gamma N_t^\delta}$.

Using equation (5) to substitute out the wage, we get

$$\frac{(\sigma-1)N_t^{\frac{1}{\sigma-1}-\delta}}{\sigma\gamma} = V_t.$$

10. Let $g = g_c$ denote the equilibrium growth rate of per-capita aggregate consumption c_t along a BGP. Since

$$c_t = \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}},$$

equation (7) implies that

$$c_t = (1-\beta_t)N_t^{-1}N_t^{\frac{\sigma}{\sigma-1}} = (1-\beta_t)N_t^{\frac{1}{\sigma-1}}.$$

Clearly, $\beta_t \in (0,1)$ must be constant along any BGP (more in TD), so that the previous equation implies

$$g = \frac{1}{\sigma-1}g_N.$$

Moreover, we get from the wage equation (5) that

$$g_w = \frac{1}{\sigma-1}g_N = g. \tag{10}$$

11. Recall that the knowledge accumulation equation writes

$$\frac{dN_t/dt}{N_t} = \gamma N_t^{\delta-1}\beta_t L_t.$$

Along any BGP, the left-hand side of this equation is equal to g_N, so that the right-hand side must be constant over time. It follows that

$$(1-\delta)g_N = n,$$

where n is the growth rate of the population. Using equation (10), we get

$$g = \frac{n}{(1-\delta)(\sigma-1)}, \tag{11}$$

as was to be shown.

12. The equilibrium growth rate in this economy is increasing with the rate of population growth n, which was not the case in the model developed in the class notes. Moreover, positive population growth is necessary here to generate long-run growth, since $g = 0$ if $n = 0$.

This is because the externality from the stock of knowledge to the innovation process (the N_t^δ term) features decreasing returns to scale since $\delta < 1$. Hence, spillover effects and knowledge accumulation are not sufficient to sustain long-run growth by themselves. It is continuous population growth that, by allowing to devote more and more workers to R&D and fuelling innovation, generate sustained growth here. This is why the model is dubbed 'semi-endogenous': without exogenous population growth, the endogenous innovation channel is not strong enough to imply steady-state growth.

This semi-endogenous growth model improves the baseline R&D model by removing the so-called scale effect, according to which a larger population size translates into a higher equilibrium growth rate. This scale effect seems at odds with the data, as larger countries do not necessarily grow faster (though there is some debate on what are the right definitions of a country and of the time span). Additionally, the baseline endogenous growth model does not feature balanced growth with population increase: instead, the growth rate of output per capita would increase steadily over time and explode toward infinity in finite time. This is not the case in the semi-endogenous growth model studied here.

13. From the Euler equation (2), it is straightforward to obtain that along any BGP, the value of the interest rate is

$$r = g + \rho + n = \frac{n}{(1-\delta)(\sigma-1)} + \rho + n.$$

14. In this model, R&D incentives is mainly captured by the markup $(\sigma - 1)/\sigma$, since it affects the size of monopoly profits and the value of patents. Notice that $g_N = \frac{n}{(1-\delta)}$ does not depend on the level of markup. Therefore, R&D incentives do not affect the long-run growth rate of the economy. The dependence of g on σ just reflects the "love for variety" effect embedded in the Dixit-Stiglitz preferences.

2 Optimality in the R&D-based endogenous growth model

In this version of the Grossman-Helpman model, we assume that firms charge an exogenous and constant markup μ.

1. To derive the equilibrium growth rate of the economy, denoted g^C, we follow the same steps as in the lecture notes.

The Euler equation for aggregate consumption. First, we obtain a relationship between g^C and the real interest rate using the Euler equation for aggregate consumption, in the same way as in the class notes.

Using the same argument as in the first exercise, we find that the intertemporal budget constraint of the representative dynasty writes

$$W_0 + \int_0^\infty \exp\left(-\int_0^t r_s ds\right)(w_t L_t + D_t - C_t)dt \geq 0.$$

The lifetime utility maximization problem then writes

$$\max_{C_t} \quad \int_0^\infty e^{-\rho t} \ln C_t dt$$

subject to (1). This is a static program, and the first-order condition derived from a Lagrangian is

$$\frac{e^{-\rho t}}{C_t} = \lambda \exp\Big(- \int_0^t r_s ds \Big),$$

where λ denotes the Lagrange multiplier. Differentiating with respect to time yields

$$\frac{-\rho e^{-\rho t} C_t - e^{-\rho t} dC_t/dt}{C_t^2} = -r_t \lambda \exp\left(- \int_0^t r_s ds \right).$$

Rearranging, we obtain the Euler equation:

$$\frac{dC_t/dt}{C_t} = r_t - \rho. \tag{12}$$

Along any BGP, the growth rate of consumption must be constant. It thus follows from (12) that the real interest rate r_t must be constant too, so that

$$g^C = r - \rho. \tag{13}$$

This is a first equilibrium relationship between g^C and r. To solve for g^C, we need a second equation in order to solve for g^C. We will derive it from the free-entry condition in the R&D sector.

Equilibrium in the production sector. To study the equilibrium in the production sector, we use the assumption of constant markup: for all $i \in (0, N_t)$, $p_{it} = \mu w_t$. Hence, the aggregate price index is just

$$P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} d_i \right)^{\frac{1}{1-\sigma}} = \mu w_t N_t^{\frac{1}{1-\sigma}}.$$

(In this Dixit-Stiglitz framework, the aggregate price index P_t is the price level such that $P_t C_t$ is equal to aggregate expenditures on consumption. See question 4 in the next exercise.) Taking P_t as numeraire, i.e. setting $P_t = 1$, we get

$$w_t = \frac{N_t^{\frac{1}{\sigma-1}}}{\mu} \qquad \Longleftrightarrow \qquad \frac{w_t}{N_t} = \frac{N_t^{\frac{2-\sigma}{\sigma-1}}}{\mu}. \tag{14}$$

By symmetry of goods and firms, we also have

$$\begin{aligned}
\pi_t &= p_t c_t - w_t l_t \\
&= \mu w_t l_t - w_t l_t \quad \text{using that } c_t = l_t \text{ and the expression for } p_t \\
&= (\mu - 1) w_t \frac{L - L_{Rt}}{N_t} \quad \text{because } l_t = (L_t - L_{Rt})/N_t \text{ by symmetry} \\
&= (\mu - 1) \frac{w_t}{N_t} \left(L - \frac{dN_t/dt}{N_t} \right) \quad \text{using the production function for innovations.}
\end{aligned}$$

Using equation (14) to substitute out the wage rate, we get

$$\pi_t = \frac{\mu - 1}{\mu} N_t^{\frac{2-\sigma}{\sigma-1}} \left(L - \frac{dN_t/dt}{N_t} \right). \tag{15}$$

Free-entry in the R&D sector. Free-entry in the R&D sector requires equality between the value of newly-created varieties and total costs of production. At date t, L_{Rt} workers are employed in the R&D sector, representing a total wage bill of $w_t L_{Rt}$. These workers produce $L_{Rt} N_t$ new varieties, each with value V_t. Hence, the free-entry condition in the R&D sector writes

$$w_t L_{Rt} = L_{Rt} N_t V_t \qquad \Longleftrightarrow \qquad \frac{w_t}{N_t} = V_t. \tag{16}$$

Also, the Bellman equation for the value of a patent, denoted V_t, is

$$r_t V_t = \pi_t + \frac{dV_t}{dt}, \tag{17}$$

where π_t denotes date-t monopoly profits. As usual, the left-hand side of the equation is the payoff from selling the firm at price V_t and lending the cashflow on the market, while the right-hand side is the payoff from keeping the firm, receiving the dividend (here, the instantaneous profit) and being exposed to changes in the firm value. The Bellman equation just states that in equilibrium, both strategies must have the same payoff.

The BGP. We can now solve for the BGP. Letting g_V denote the growth rate of V_t, equation (17) implies that along any BGP

$$\pi_t = (r - g_V) V_t.$$

Substituting into (15) yields

$$V_t = \frac{\mu - 1}{\mu} N_t^{\frac{2-\sigma}{\sigma-1}} \frac{L - g_N}{r - g_V},$$

where g_N is the constant growth rate of the number of varieties N_t. Consolidating the above result, the expression in (14) for w_t/N_t, and the free-entry condition (16), we obtain

$$\frac{N_t^{\frac{2-\sigma}{\sigma-1}}}{\mu} = \frac{\mu - 1}{\mu} N^{\frac{2-\sigma}{\sigma-1}} \frac{L - g_N}{r - g_V} \qquad \Longleftrightarrow \qquad r - g_V = (\mu - 1)(L - g_N). \tag{18}$$

We now need to find relationships between g_V, g_N, and g_C. Equation (17) implies that along any BGP

$$g_V = g_\pi, \tag{19}$$

while equation (15) implies that

$$g_\pi = \frac{2 - \sigma}{\sigma - 1} g_N. \tag{20}$$

Additionally, we have

$$C_t = \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} = l_t N_t^{\frac{\sigma}{\sigma-1}} = \frac{L - L_{Rt}}{N_t} N_t^{\frac{\sigma}{\sigma-1}} = (L - L_{Rt}) N_t^{\frac{1}{\sigma-1}}. \tag{21}$$

Moreover, $L_{Rt} = (dN_t/dt)/N_t$ must be constant along any BGP, so equation (21) implies that

$$g_C = \frac{1}{\sigma - 1} g_N. \tag{22}$$

Using equations (19), (20), (22) to substitute out g_V and g_N in (18), we obtain

$$r - (2 - \sigma) g_C = (\mu - 1)[L - (\sigma - 1) g_C]. \tag{23}$$

Equations (13) and (23) form a system of two equations in two unknowns, which we can solve for g_C (and r, although we don't need it here). We obtain

$$g_C = \frac{(\mu - 1)L - \rho}{\mu(\sigma - 1)}, \tag{24}$$

which is the equilibrium growth rate of the economy along a BGP. It is increasing with the markup:

$$\frac{dg_C}{d\mu} = \frac{L + \rho}{\mu^2(\sigma - 1)} > 0 \text{ if } \sigma > 1.$$

This reflects two economic forces. First, a higher markup raises both profits and the value of a patent, thereby boosting R&D. A more subtle effect goes through the labor market. A higher markup lowers the real wage in the production sector, as illustrated by (14). This depresses final demand and shifts labor from the production sector to the R&D sector, again boosting innovation.

2. We now solve a planning problem to compute the social optimum. To write the maximization program, it is useful to obtain a simple expression for aggregate consumption. Notice that both

$$C_t = \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$

and the production functions

$$c_{it} = l_{it}$$

are symmetric with respect to the varieties. Hence, the social optimum must be such that

$$c_{it} = c_t = l_t = \frac{L - L_{Rt}}{N_t}$$

for all varieties $i \in (0, N_t)$. Hence, the same argument as in (21) applies and we have

$$C_t = (L - L_{Rt}) N_t^{\frac{1}{\sigma-1}}.$$

The planning problem then writes

$$\max_{L_{Rt}, N_t} \quad \int_0^\infty e^{-\rho t} \ln\left[(L - L_{Rt}) N_t^{\frac{1}{\sigma-1}}\right] dt$$
$$\text{s.t.} \quad \frac{dN_t}{dt} = N_t L_{Rt}.$$

Letting λ_t denote the costate variable, the corresponding Hamiltonian is

$$H(L_{Rt}, N_t, \lambda_t, t) = e^{-\rho t}\left[\frac{1}{\sigma-1}\ln N_t + \ln(L - L_{Rt})\right] + \lambda_t N_t L_{Rt}.$$

Neglecting the transversality condition, optimality with respect to the control variable L_{Rt} and the state variable N_t requires

$$\frac{e^{-\rho t}}{L - L_{Rt}} = \lambda_t N_t,$$
$$\frac{e^{-\rho t}}{(\sigma-1)N_t} + \lambda_t L_{Rt} = -\frac{d\lambda_t}{dt}.$$

Defining $\mu_t := e^{\rho t}\lambda_t$, we obtain

$$\frac{1}{L - L_{Rt}} = \mu_t N_t, \tag{25}$$
$$\frac{1}{(\sigma-1)N_t} + \mu_t L_{Rt} = \rho\mu_t - \frac{d\mu_t}{dt}. \tag{26}$$

Along a BGP, we know that L_{Rt} must be constant and equal to $g_N^\star$, where the star highlights that we are along the optimal trajectory. Hence, equation (25) implies that

$$g_\mu^\star = -g_N^\star.$$

Moreover, consolidating equations (25) and (26) yields

$$\frac{L - L_{Rt}}{\sigma - 1} + L_{Rt} = \rho - \frac{d\mu_t/dt}{\mu_t},$$

which, together with $L_R = g_N^\star$, implies that

$$g_N^\star = L - (\sigma-1)\rho.$$

Eventually, along any BGP we have that

$$C_t = (L - g_N^\star) N_t^{\frac{1}{\sigma-1}},$$

so that the optimal growth rate of the economy is

$$g_C^\star = \frac{g_N^\star}{\sigma-1} = \frac{L}{\sigma-1} - \rho. \tag{27}$$

A minor difficulty that arises from this expression is that if agents are too impatient, that is if ρ is too large, aggregate consumption will decrease over time. However, this is not possible here.

Indeed, if $g_C^\star < 0$ then $g_N^\star < 0$ too, which would imply in turn $L_R < 0$ in the long run, but a workforce cannot be negative. For the model to make sense, we thus need to assume that the parameters are such that $g_C^\star > 0$ and $L_R > 0$.

3. The optimal markup $\mu^\star$ equalizes the equilibrium growth rate of consumption given in (24) with its optimal counterpart given in (27). Hence, it verifies

$$\frac{(\mu^\star - 1)L - \rho}{\mu^\star(\sigma - 1)} = \frac{L}{\sigma - 1} - \rho \qquad \Longleftrightarrow \qquad \mu^\star = \frac{L + \rho}{\rho(\sigma - 1)}.$$

Here also, we need to restrict parameters to ensure that $\mu^\star > 1$. Otherwise, firms would make negative profits in equilibrium and would leave the market.

If $\mu > \mu^\star$, we have $g_C > g_C^\star$ and the growth rate of consumption is too high in the competitive equilibrium, meaning that current consumption is too low. This is because a high markup implies that R&D is too profitable compared to the social optimum, in which case consumption is necessarily too low because not enough workers are employed in the production sector. On the other hand, if $\mu < \mu^\star$, $g_C < g_C^\star$ and the equilibrium growth rate of consumption is too low because not enough resources are devoted to R&D.

There are two sources of inefficiencies in the decentralized equilibrium: monopolistic competition in the good sector implies that prices are too high, while externalities from the stock of knowledge on the production of new varieties is not internalized by private agents. The first inefficiency boosts R&D: monopolists make higher profits than competitive firms would, and therefore the value of a patent is too high. The second inefficiency, on the other hand, means that there is not enough R&D, since private returns are smaller than public returns. The optimal markup that we have computed balances those two effects to restore optimality.

3 The augmented Solow model with only accumulable factors

1. Given the production function and the accumulation equations for physical and human capital, it is straightforward to derive that

$$\frac{dK_t/dt}{K_t} = s_K A(H_t/K_t)^{1-\alpha},$$
$$\frac{dH_t/dt}{H_t} = s_H A(K_t/H_t)^{\alpha}.$$

Integrating forward yields

$$K_t = K_0 \exp\left(s_K A \int_0^t (H_\tau/K_\tau)^{1-\alpha} d\tau\right),$$
$$H_t = H_0 \exp\left(s_H A \int_0^t (K_\tau/H_\tau)^{\alpha} d\tau\right).$$

It is then immediate that changing initial conditions from (K_0, H_0) to $(\lambda K_0, \lambda H_0)$, with $\lambda > 0$, changes the economy's trajectory from (K_t, H_t) to $(\lambda K_t, \lambda H_t)$.

2. First, let us characterize a balanced growth path of this economy. From the accumulation equations $dK_t/dt = s_K Y_t$ and $dH_t/dt = s_H Y_t$, it must be that Y_t, K_t, and H_t all grow at the same

rate along any BGP. Then, notice that

$$\frac{dK_t/dt}{K_t} = \frac{dH_t/dt}{H_t} \quad \Longleftrightarrow \quad s_K A\left(\frac{H_t}{K_t}\right)^{1-\alpha} = s_H A\left(\frac{K_t}{H_t}\right)^{\alpha} \quad \Longleftrightarrow \quad \frac{H_t}{K_t} = \frac{s_H}{s_K},$$

so $x_t = H_t/K_t$ must be constant along any BGP. Convergence then follows from the fact that

$$\frac{dx_t/dt}{x_t} = \frac{dH_t/dt}{H_t} - \frac{dK_t/dt}{K_t} = s_H A x_t^{-\alpha} - s_K A x_t^{1-\alpha} = \frac{A}{s_K} x^{-\alpha}\left(\frac{s_H}{s_K} - x_t\right). \tag{28}$$

Since x_t increases over time iff $x_t < s_H/s_K$ and decreases iff $x_t > s_H/s_K$, we have $x_t \to s_H/s_K$ and there is indeed convergence to a BGP.

3. Along the BGP, $H_t/K_t = s_H/s_K$. It follows that the growth rate of the economy is given by

$$s_K A\left(\frac{s_H}{s_K}\right)^{1-\alpha} = s_H A\left(\frac{s_K}{s_H}\right)^{\alpha} = A s_K^{\alpha} s_H^{1-\alpha}.$$

The exponents on the saving rates in this weighted geometric mean exactly match those on the respective factors in the production function. Hence, if production becomes more intensive in one specific factor, the long-run growth rate of the economy will depend more on the saving rate associated with that particular factor.

4. According to equation (28), a necessary and sufficient condition for the economy to be on a BGP from date 0 on is that

$$\frac{H_0}{K_0} = \frac{s_H}{s_K}.$$

It is sufficient because it implies $dx_t/dt = 0$ for all t, in which case K_t, H_t, and thus Y_t grow at the same constant rate. It is necessary because otherwise $dx_t/dt \neq 0$ at date 0, which is not compatible with a BGP.