Macroeconomics II – Problem Set 5

Solutions

TAs 2016: Yue Fei and George Lukyanov

1 Two-sided limited commitment: a simple example

i. Necessary condition for perfect risk-sharing to be sustainable for the first household is

$$U(\frac{1}{2}) \geq D^1(L) \qquad \text{and} \qquad U(\frac{1}{2}) \geq D^1(H)$$

where $U()$ is the utility function and $D^1(L)$ and $D^1(H)$ are the value of autarchy in state L and H respectively.

$D^1(L)$ and $D^1(H)$ can be recovered from the following equation

$$D^1(y) = u(y)(1-\beta\Pi)^{-1} \tag{1.1}$$

Similarly, Necessary condition for perfect risk-sharing to be sustainable for the second household is

$$U(\frac{1}{2}) \geq D^2(L) \qquad \text{and} \qquad U(\frac{1}{2}) \geq D^2(H)$$

where $D^2(L)$ and $D^2(H)$ can be recovered from the following equation

$$D^2(y) = u(1-y)(1-\beta\Pi)^{-1}$$

The original equation system behind (1.1) is

$$D^1(L) = u(L) + \beta\Sigma_{y'}\pi(y'|L)D^1(y')$$

$$D^1(H) = u(H) + \beta\Sigma_{y'}\pi(y'|H)D^1(y')$$

As $\beta \to 0$, this inequality cannot be satisfied because $U(1/2) < U(H) = D^j(H)$, $j = 1, 2$.

As the persistence increases (this is,$\pi(H|H) \to 1$, $\pi(L|L) \to 1$, $\pi(H|L) \to 0$, $\pi(L|H) \to 0$, i.e., $\Pi \to I$), $D^j(H) \to U(H)/(1-\beta) > U(1/2)/(1-\beta)$, $j = 1, 2$. So this condition is violated.

ii. A planner wants to solve the following problem

$$\max_{c^j(\theta^t);n^j(\theta^t)} \Sigma_{j=1}^J \lambda_0^j \Sigma_{\theta^t}\beta^t\pi(\theta^t)u(c^j(\theta^t))$$

subject to

$$\Sigma_{j=1}^J \lambda_0^j c^j(\theta^t) \leq \Sigma_{j=1}^J \lambda_0^j y^j(\theta^t), \qquad \forall\theta^t$$

and a limited commitment constraint

$$\Sigma_{\theta^\tau|\theta^t}\beta^{\tau-t}\frac{\pi(\theta^\tau)}{\pi(\theta^t)}u(c^j(\theta^t) \geq D^j(\theta^t), \qquad \forall j, \theta^t$$

Constructing the Lagrangian,

$L = \Sigma_{j=1}^J \lambda_0^j \Sigma_{\theta^t}\beta^t\pi(\theta^t)u(c^j(\theta^t))$
$+ \Sigma_{\theta^t}\lambda(\theta^t)\{\Sigma_{j=1}^J\lambda_0^j y^j(\theta^t) - \Sigma_{j=1}^J\lambda_0^j c^j(\theta^t)\}$
$+ \Sigma_{j=1}^J\Sigma_{\theta^t}\phi^j(\theta^t)\{\Sigma_{\theta^\tau}\beta^\tau\pi(\theta^\tau)u(c^j(\theta^t)) - \beta^t\pi(\theta^t)D^j(\theta^t)\}$

We introduce the concept of a cumulative multiplier. Define Φ to be such that $\Phi^j(\theta^t) = \Phi^j(\theta^{t-1}) + \phi^j(\theta^t)$, $\forall t \geq 1$ and $\Phi^j(\theta^0) = \phi^j(\theta^0)$.

Whenever the agent strictly prefers to stay in the contract, the original multiplier takes on a value of zero. Whenever the agent is indifferent between staying and walking away, the multiplier becomes positive. Thus, the cumulative multiplier goes up (is non-decreasing) over time. When the limited commitment constraint is binding, the multiplier gets pumped up, when it is not binding, the cumulative mul-tiplier stays the same.

So we could get a modified Lagrangian,

$L = \Sigma_{j=1}^J(\lambda_0^j + \Phi^j(\theta^t))\Sigma_{\theta^t}\beta^t\pi(\theta^t)u(c^j(\theta^t))$
$+ \Sigma_{\theta^t}\lambda(\theta^t)\{\Sigma_{j=1}^J\lambda_0^j y^j(\theta^t) - \Sigma_{j=1}^J\lambda_0^j c^j(\theta^t)\}$
$+ \Sigma_{\theta^t}(\Phi^j(\theta^t) - \Phi^j(\theta^{t-1}))\beta^t\pi(\theta^t)D^j(\theta^t)\}$

iii. In an endowment economy, the value of being autarky is

$$D^j(\theta^t) = \Sigma_{\tau=t}^\infty\Sigma_{\theta^\tau|\theta^t}\beta^{\tau-t}\pi(\theta^\tau|\theta^t)u(y_\tau^j(\theta^\tau))$$

The first order condition for consumption for household j in node θ^t is

$$(\lambda_0^j + \Phi^j(\theta^t))\beta^t\pi(\theta^t))u_c(c^j(\theta^t)) = \lambda_0^j\lambda(\theta^t)$$

Defining

$$\xi^j(\theta^t) = \frac{\lambda_0^j + \Phi^j(\theta^t)}{\lambda_0^j}$$

The ratio of first order conditions for two households i and j is,

$$\frac{u_c(c^i(\theta^t))}{u_c(c^j(\theta^t))} = \frac{\xi^j(\theta^t)}{\xi^i(\theta^t)}$$

The complementarity slackness conditions are

$$\lambda(\theta^t)\{\Sigma_{j=1}^J\lambda_0^j y^j(\theta^t) - \Sigma_{j=1}^J\lambda_0^j c^j(\theta^t)\} = 0$$

$$\phi^j(\theta^t)\{\Sigma_{\theta^\tau}\beta^\tau\pi(\theta^\tau)u(c^j(\theta^t)) - \beta^t\pi(\theta^t)D^j(\theta^t)\} = 0$$

Conjecture a risk-sharing rule

$$c^i(\theta^t) = \frac{\xi^i(\theta^t)^{\frac{\theta-1}{\theta}}}{\Sigma_{j=1}^J \xi^j(\theta^t)^{\frac{\theta-1}{\theta}}} \Sigma_{j=1}^J y^j(\theta^t)$$

It is easy to verify that the risk sharing rule satisfies first order conditions and market clearing. History dependence and time-varying weights are signatures of enforce-ment and private information problems.

Typically, a households participation constraint binds when it switches to a high income state, if the endowment is persistent. In those states of the world, the value of autarchy is high and agents are tempted to default. When agents switch to these states, their multiplier increases, raising their consumption shares, depending on how severely other agents are constrained (captured by the denominator of the risk sharing rule).

Amnesia property: A households consumption share decreases as long as it does not switch to a state with a binding constraint, but when it does, its consumption share increases to some cutoff level that does not depend on the history(θ^t)if the endowment process is first-order Markov.

Proof:

The first part follows from the risk sharing rule and the fact that $\xi^j(\theta^t)$ is a non-decreasing process for all j. The second part follows from the complementary slackness condition, which says that, when constraint binds

$$\Sigma_{\theta^\tau} \beta^\tau \pi(\theta^\tau) u(c^j(\theta^t)) - \beta^t \pi(\theta^t) D^j(\theta^t) = 0$$

Now, if y is first order Markov, then $D^j(\theta^t)$ only depends on θ^t. This implies $\{c^j(\theta^\tau)\}_{\tau=1}^t$ cannot depend on θ^t, only on θ_t.

iv. We use consumption weights as state variables instead of cumulative multipliers because we want to use stationary state variables.

The consumption share of household i in period t , node θ^t is defined as

$$\omega_t^i = \frac{\xi^i(\theta^t)^{\frac{\theta-1}{\theta}}}{\Sigma_{j=1}^J \xi^j(\theta^t)^{\frac{\theta-1}{\theta}}}$$

When the household i does NOT switch to a state with a binding constraint, her consumption share next period is

$$\omega_{t+1}^i = \omega_t^i \frac{\xi^i(\theta^t)^{\frac{\theta-1}{\theta}}}{\Sigma_{j=1}^J \xi^j(\theta^{t+1})^{\frac{\theta-1}{\theta}}}$$

When the household i does switch to a state with a binding constraint, her consumption share next period is

$$\omega_{t+1}^i = \omega_t^i \frac{\xi^i(\theta^t)^{\frac{\theta-1}{\theta}}}{\Sigma_{j=1}^J \xi^j(\theta^{t+1})^{\frac{\theta-1}{\theta}}} \frac{\xi^i(\theta^{t+1})^{\frac{\theta-1}{\theta}}}{\xi^i(\theta^t)^{\frac{\theta-1}{\theta}}}$$

At the start of next period, we compare the agents consumption share in the previous period ω_{t-1}^i to the cutoff value for the current state of the world: $\underline{\omega}^1(\theta)$. If the weight exceeds the cutoff weight, the consumption weight is left unchanged. If the consumption weight is smaller than the cutoff weight, the agents actual consumption is the cutoff weight. The cutoff rule is determined such that the constraint binds exactly. Notice, in two agent case, we have three cases 1) Agent 1's constraint is binding (the lower cutoff); 2) Agent 2's constraint is binding (the higher cutoff); 3) Neither is binding.

We solve for the cutoff values,

$$D^1(y) = U^1(\underline{\omega}^1(y), y =)u(\underline{\omega}^1) + \beta\Sigma_{y'}\pi(y'|y)U^1(\omega')$$
$$D^2(y) = U^2(\overline{\omega}^1(y), y =)u(1 - \overline{\omega}^1) + \beta\Sigma_{y'}\pi(y'|y)U^2(\omega')$$

where ω' in the next period is found by applying the following rule,

$$\text{if} \quad \underline{\omega}^1 < \omega^1 < \overline{\omega}^1, \omega^{1\prime} = \omega^1$$
$$\text{if} \quad \underline{\omega}^1 > \omega^1, \quad \omega^{1\prime} = \underline{\omega}^1$$
$$\text{if} \quad \omega^1 > \overline{\omega}^1, \quad \omega^{1\prime} = \overline{\omega}^1$$

v. Perfect risk sharing is feasible when the intersection of the two intervals $(\underline{\omega}^1, \overline{\omega}^1)$ is non-empty. If it is non-empty, it contains a consumption share of a half.

When perfect risk sharing is not feasible, agent 1's consumption values $c^1(\theta_1)$ and $c^1(\theta_2)$ takes the value of ($\underline{\omega}^1$ and $\overline{\omega}^1$ (Because the total endowment is 1, the consumption share is exactly the consumption level.)

vi. For this part see the code `LC2.m`.

6.a Solving the system for the benchmark parameters we get:

$$c(H) = 0.5375 < y(H)$$
$$c(L) = 0.4625 > y(L)$$

So there is some risk sharing, but not perfect.

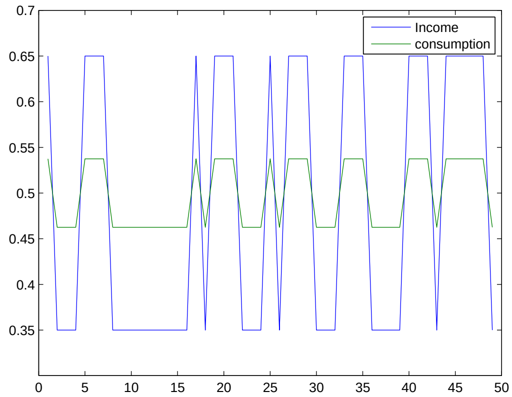
6.b A possible path for consumption and income is shown in Figure **??**. The graph shows the imperfect risk sharing. The initial consumption of agent 1 may lie in $[0.5375, 0.65]$ (a consumption below 0.5375 would make agent 1 go to autharky and a consumption above 0.65 will make agent 2 go to autharky), and will remain there until receiving a bad shock. After that, consumption of these agents will jump between $[c(H), c(L)]$ as shown in the graph.

6.c For $\beta = 0.8$ no risk sharing is possible. However, for $\beta = 0.99$ the conditions for perfect risk sharing are met, therefore perfect risk sharing is possible. The more patient are the agents, more risk sharing is possible since they value the future more and because of risk aversion, remaining in the contract is more valuable.

6.d When $\pi(H|H) = 0.95$ the income process is very persistent and risk sharing is not possible. The more persistent the income process, the more difficult is risk sharing. The intuition is simple, in the extreme if you know you are rich for sure and will never receive a transfer from the rest, the contract represents a loss of utility to you and you prefer to renege of it.

On the other hand when $\pi(H|H) = 0.55$, when persistence is low, the contract is valuable since it will allow to smooth consumption. In this case perfect risk sharing is possible.

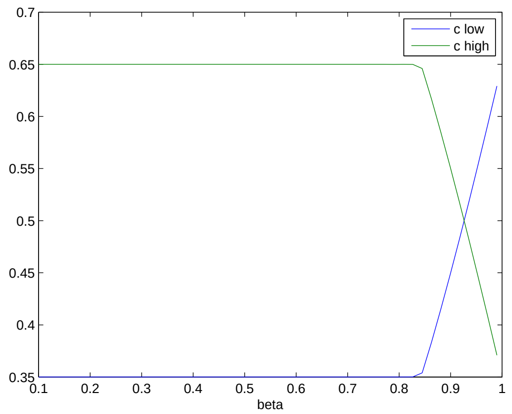
Figure 1: Dynamics of Consumption



6.e In this case, agents are risk neutral, so they are indifferent between a risky path of consumption and a non risky one which delivers the same expected value. Because they are indifferent, the value of remaining in a risk sharing contract is the same as autharky. Therefore, there is an incentive to always renege and no contract is possible.

6.f In this case agents are very risk averse. As mentioned before, if β is very low, no risk sharing is possible. On the other hand, if beta is high enough even perfect risk sharing is possible. The lowest value for β such that some risk sharing is possible is close to 0.81. Figure 2 shows the results.

Figure 2: Dynamics of Consumption



2 Limited Commitment and Hidden Storage

This problem adds storage to the standard endowment economy with limited commitment. Storage is inefficient, so it shouldn't be used as part of the contract, but it does increase autarky values, and hence reduces risk-sharing. In the extreme case, where r is close to 1, it will completely crowd out voluntary transfers.

i. The first-best allocation $\{c_t^A, c_t^B\}$ maximizes

$$\sum_{t=0}^{\infty} \beta^t \left[\log\left(c_t^A\right) + \mu \log\left(c_t^B\right)\right] \tag{2.1}$$

subject to the per-period resource constraint:

$$c_t^A + c_t^B \leq 1 \tag{2.2}$$

where μ denotes type B's Pareto weight. The optimal solution to this problem is stationary over time and solves the first order condition:

$$\frac{1}{c_t^A} = \frac{\mu}{1 - c_t^A} \tag{2.3}$$

In the symmetric case with $\mu = 1$, we have $c_t^A = 1/2$.

ii. Let $v_H(k)$ (correspondingly, $v_L(k)$) define the autarky value of an agent who enter a high (correspondingly, low) income period with storage k. $v_H(k)$ and $v_L(k)$ should satisfy the pair of Bellman equations:

$$v_H(k) = \max_{k' \geq 0} \left\{\log(y + rk - k') + \beta v_L(k')\right\} \tag{2.4a}$$

$$v_L(k) = \max_{k' \geq 0} \left\{\log(1 - y + rk - k') + \beta v_H(k')\right\} \tag{2.4b}$$

Now, the second best contracting problem with voluntary transfers consists in finding allocations $\{c_t^A, c_t^B\}$ and storage levels $\{k_t^A, k_t^B\}$ that maximize

$$\sum_{t=0}^{\infty} \beta^t \left[\log\left(c_t^A\right) + \mu \log\left(c_t^B\right)\right] \tag{2.5}$$

subject to the per-period resource constraints

$$c^A + c^B \leq 1 + r(k^A + k^B) - (k^{A'} + k^{B'}) \tag{2.6}$$

the participation constraints for A:

$$\sum_{t=s}^{\infty} \beta^{t-s} \log(c_t^A) \geq \begin{cases} v_H(k_s), & \text{if } A \text{ receives a high endowment in period } s \\ v_L(k_s), & \text{if } A \text{ receives a low endowment in period } s \end{cases} \tag{2.7a}$$

and similar participation constraints for B:

$$\sum_{t=s}^{\infty} \beta^{t-s} \log(c_t^B) \geq \begin{cases} v_H(k_s), & \text{if } B \text{ receives a high endowment in period } s \\ v_L(k_s), & \text{if } B \text{ receives a low endowment in period } s \end{cases} \tag{2.7b}$$

iii. At the first-best allocation, storage is not used, and consumption equals 1/2 in all periods, so we simply need to check that

$$\frac{\log(1/2)}{1-\beta} \geq v_H(0) \quad \text{and} \quad \frac{\log(1/2)}{1-\beta} \geq v_L(0)$$

Since $v_L(0) < v_H(0)$, only the first constraint is relevant. Now, for this we need to find $v_H(0)$. It is immediate that

$$v_H(0) = \frac{1}{1-\beta^2} \max_{k'} \Big\{ \log(y - k') + \beta \log(1 - y + rk') \Big\} \tag{2.8}$$

This expression emerges, because at the solution to the above pair of Bellman equations, storage is zero entering a high-income period, but positive entering low-income periods. One immediately observes that for any $k' \in [0, y]$, $1 - y + rk'$ may be made arbitrarily small, if y is close to 1, and r is sufficiently close to zero. But then the above participation constraint holds.

iv. First, notice that storage isn't used by the optimal contract because it (i) tightens current resources in exchange for future resources at a loss (undesirable due to discounting), and (ii) it tightens the participation constraints. Thus, we can ignore storage within the contract, set $k_s = 0$ for all periods, and just focus on the participation constraints. Now, the binding constraint is for agents with high endowments, which leads to the constraint

$$v_H(0) = \frac{1}{1-\beta^2} \Big\{ \log(c) + \beta \log(1 - c) \Big\}$$

which defines the solution for c. A simple comparative static emerges: since $v_H(0)$ is increasing in r, it is the case that a higher return on storage increases the value of the outside option, and hence reduces the scope for consumption-smoothing using voluntary transfers.

v. As $r \to 1$, the agent comes closer and closer to being able to replicate any split $c < y$ with an appropriate choices of storage. This reduces, and in the limit eliminates, the ability to smooth consumption with transfers; they are thus crowded out by storage. When $r = 1$, the crowding out is complete, and no voluntary transfers are implementable.