Macroeconomics I—Problem Set 5
Dynamic Stochastic Economy with Complete Markets, Planner's Problem

Christian Hellwig

TAs 2013: George Lukyanov, Harry Di Pei

TAs 2014: Constance de Soyres and Francois de Soyres

TAs 2015: Yue Fei and Lan Lan

November 9, 2015

1 Several Simple Extensions

Recall that for the standard case seen in class, the planner's problem is to choose the sequence of consumption allocations, labor supply and capital stocks, $\{c_j(\cdot), n_j(\cdot), K(\cdot)\}$, which would solve[1]

$$\max_{\{c_j(\cdot),n_j(\cdot),K(\cdot)\}} \sum_{j=1}^{J} \psi_j \sum_{t,s^t} \beta^t \pi(s^t) u^j(c_j(s^t), n_j(s^t)) \tag{1.1}$$

subject to the sequence of aggregate resource constraints,

$$C(z^t) + K(z^t) \leq A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta)K(z^{t-1}), \quad \forall z^t \tag{1.2}$$

where

$$C(z^t) = \sum_{j=1}^{J} \sum_{h^t} \mu_j \pi(h^t) c_j(z^t, h^t)$$

denotes aggregate consumption, while

$$N(z^t) = \sum_{j=1}^{J} \sum_{h^t} \mu_j \pi(h^t) \theta^j(z^t, h^t) n_j(z^t, h^t)$$

is the total supply of efficient labor in state z^t.

The associated Lagrangian is given by

$$\begin{aligned} \mathcal{L} = & \sum_{j=1}^{J} \psi_j \sum_{t,s^t} \beta^t \pi(s^t) u^j(c_j(s^t), n_j(s^t)) + \\ & \sum_{z^t} \lambda(z^t) \left[A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta)K(z^{t-1}) - K(z^t) - C(z^t) \right] \end{aligned} \tag{1.3}$$

[1] The sequence $(\psi_1, \ldots, \psi_J)$ denotes the weights that the planner attaches to well-being of class-j households, whereas $(\mu_1, \ldots, \mu_J)$ stands for their measure in the population.

with the corresponding FOC's (for $c_j(\cdot)$, $n_j(\cdot)$ and $K(\cdot)$, respectively):

$$\psi_j \beta^t \pi(s^t) u_c^j(c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t)$$

$$-\psi_j \beta^t \pi(s^t) u_n^j(c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t) A(z^t) \theta^j(s^t) f_n(K^*(z^{t-1}), N^*(z^t))$$

$$\lambda(z^t) = \sum_{z^{t+1} \succ z^t} \lambda(z^{t+1}) \left\{ A(z^{t+1}) f_K(K^*(z^t), N^*(z^{t+1})) + 1 - \delta \right\}$$

Let us summarize the qualitative results one may derive from the above conditions:

1. Efficient allocation equates each household's marginal rate of substitution between consumption and leisure to his marginal product of labor;
2. Household-specific productivity shocks only affect the household's marginal disutility of effort, but not the household's marginal utility of consumption;
3. Inter-temporal marginal rate of substitution is equalized across all households;
4. The efficient investment level satisfies the inter-temporal Euler Equation for each household.

Now we introduce several twists and see how it affects the above conclusions:

- Suppose the '*taste*' of type j household varies with s^t: type j's utility is given by $u^j(\chi^j(s^t)c(s^t), n(s^t))$, where $\chi^j(s^t)$ is a preference shock.

  In that case, the Lagrangian would be

$$\mathcal{L} = \sum_{j=1}^{J} \psi_j \sum_{t,s^t} \beta^t \pi(s^t) u^j(\chi^j(s^t) c_j(s^t), n_j(s^t)) + \sum_{z^t} \lambda(z^t) \left[A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta) K(z^{t-1}) - K(z^t) - C(z^t) \right] \tag{1.4}$$

  The FOC's for $c_j(\cdot)$ and $n_j(\cdot)$ get modified to

$$\psi_j \beta^t \pi(s^t) \chi^j(s^t) u_c^j(\chi^j(s^t) c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t)$$

$$-\psi_j \beta^t \pi(s^t) u_n^j(\chi^j(s^t) c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t) A(z^t) \theta^j(s^t) f_n(K^*(z^{t-1}), N^*(z^t))$$

  while the FOC for $K(\cdot)$ stays intact.

  Dividing the second condition by the first, we arrive at the following counterpart to Result 1: each household's marginal rate of substitution between consumption and leisure has to be equal to his marginal product of labor, *multiplied by the taste shock*:

$$\chi^j(s^t) \theta^j(s^t) A(z^t) f_n(K^*(z^{t-1}), N^*(z^t)) = -\frac{u_n^j(\chi^j(s^t) c_j^*(s^t), n_j^*(s^t))}{u_c^j(\chi^j(s^t) c_j^*(s^t), n_j^*(s^t))} \tag{1.5}$$

Now the differences in the marginal rates of substitution are reflected *both* in the heterogeneity in skills and in the heterogeneity in tastes: what matters for the planner is how much the return of an additional working hour is *valued* by the household, – which is captured by the $\chi^j(\cdot)$ term. Those with larger $\chi^j(s^t)$ have a higher marginal valuation for consumption, hence their optimal point should correspond to higher MRS between consumption and leisure.[2]

Result 2 will not hold in general. As long as household-specific productivity shocks affect χ^j, the marginal utility of consumption as well as the marginal disutility of effort will be affected. Result 2 will hold only in a specific case: when the household-specific shock affects his productivity but not his χ^j, only marginal disutility of effort will be affected, while the marginal utility of consumption will not. Formally, for all $h^t, \tilde{h}^t$ with $\chi^j(z^t, h^t) = \chi^j(z^t, \tilde{h}^t)$, we have

$$\chi^j(s^t)u_c^j(\chi^j(s^t)c_j^*(s^t), n_j^*(s^t)) = \frac{\mu_j}{\psi_j}\frac{\lambda(z^t)}{\beta^t\pi(z^t)} = \chi^j(\tilde{s}^t)u_c^j(\chi^j(\tilde{s}^t)c_j^*(\tilde{s}^t), n_j^*(\tilde{s}^t)) \tag{1.6}$$

where $\tilde{s}^t = (z^t, \tilde{h}^t)$. On the other hand, since we must have

$$-u_n^j(\chi^j(s^t)c_j^*(s^t), n_j^*(s^t)) = \frac{\mu_j}{\psi_j}\theta^j(s^t)\frac{\lambda(z^t)}{\beta^t\pi(z^t)}A(z^t)f_n(K^*(z^{t-1}), N^*(z^t)) \tag{1.7}$$

for all s^t, the household's marginal disutility of effort will be affected whenever the shock affects the marginal productivity, i.e. for all $s^t, \tilde{s}^t$ with $\theta^j(s^t) \neq \theta^j(\tilde{s}^t)$.

Concerning Result 3, we claim that differences in the households' inter-temporal marginal rates of substitution have to reflect the differences in the taste shocks: for any $j = 1, \ldots, J$ and any s^t, we must have

$$\beta^t\pi(z^t)\frac{\chi^j(s^t)u_c^j(\chi^j(s^t)c_j^*(s^t), n_j^*(s^t))}{\chi^j(s^0)u_c^j(\chi^j(s^0)c_j^*(s^0), n_j^*(s^0))} = \frac{\lambda(z^t)}{\lambda(z^0)} \tag{1.8}$$

so that given z^t, for all i and j it holds that

$$\frac{u_c^j(\chi^j(s^t)c_j^*(s^t), n_j^*(s^t))}{u_c^j(\chi^j(s^0)c_j^*(s^0), n_j^*(s^0))} = \frac{u_c^i(\chi^i(s^t)c_i^*(s^t), n_i^*(s^t))}{u_c^i(\chi^i(s^0)c_i^*(s^0), n_i^*(s^0))} \cdot \frac{\chi^i(s^t)/\chi^j(s^t)}{\chi^i(s^0)/\chi^j(s^0)} \tag{1.9}$$

As for Result 4, the efficient level will now satisfy the following modified version of the inter-temporal Euler Equation:

$$1 = \sum_{z^{t+1}\succ z^t}\beta\pi(z^{t+1}|z^t)\frac{\chi^j(s^{t+1})u_c^j(\chi^j(s^{t+1})c_j^*(s^{t+1}), n_j^*(s^{t+1}))}{\chi^j(s^t)u_c^j(\chi^j(s^t)c_j^*(s^t), n_j^*(s^t))}\times$$

$$\left\{A(z^{t+1})f_K(K^*(z^t), N^*(z^{t+1})) + 1 - \delta\right\}$$

It can be seen from the above equation that an increase in the current taste shock $\chi^j(s^t)$ for some j will tend to reduce aggregate investment, *ceteris paribus*, since in that case

[2]That could be easily seen via the indifference curves in the (l, c) space.

individuals have a higher marginal valuation for the current consumption.[3] Similar reasoning tells us that higher $\chi^j(s^{t+1})$ will tend to raise investment (and so reduce current consumption), since future consumption starts to be valued more.

- Now suppose that the taste shock $\chi^j(s^t)$ multiplicatively affects the household's disutility from work. The Lagrangian becomes

$$\mathcal{L} = \sum_{j=1}^{J} \psi_j \sum_{t,s^t} \beta^t \pi(s^t) u^j(c_j(s^t), \chi^j(s^t) n_j(s^t)) + \tag{1.10}$$
$$\sum_{z^t} \lambda(z^t) \left[A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta) K(z^{t-1}) - K(z^t) - C(z^t) \right]$$

while the FOC's for consumption and labor are, respectively[4]

$$\psi_j \beta^t \pi(s^t) u_c^j(c_j^*(s^t), \chi^j(s^t) n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t)$$

$$-\psi_j \beta^t \pi(s^t) \chi^j(s^t) u_n^j(c_j^*(s^t), \chi^j(s^t) n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t) A(z^t) \theta^j(s^t) f_n(K^*(z^{t-1}), N^*(z^t))$$

while the FOC for $K(.)$ stays intact.
Concerning Result 1, the household j's MRS between consumption and leisure will be equal to his marginal product of labor, *divided* by the taste shock $\chi^j(s^t)$:

$$\frac{\theta^j(s^t)}{\chi^j(s^t)} A(z^t) f_n(K^*(z^{t-1}), N^*(z^t)) = -\frac{u_n^j(c_j^*(s^t), \chi^j(s^t) n_j^*(s^t))}{u_c^j(c_j^*(s^t), \chi^j(s^t) n_j^*(s^t))} \tag{1.11}$$

The interpretation is very similar to the one we had before: what matters for the planner is the "effective" marginal product of labor, normalized by the disutility from work: it is still true that, conditional on $\theta^j(s^t)/\chi^j(s^t)$ and on the aggregate state z^t, the marginal rates of substitution between consumption and leisure have to be equalized across households.

Just as above, Result 2 does not hold in general. Yet, all the shocks that affect θ^j but not χ^j will have no impact on the household's marginal utility of consumption, u_c^j, and will only affect $-u_n^j$. Furthermore, if $u^j(c_j(s^t), \chi^j(s^t) n_j(s^t))$ is separable in $c_j(\cdot)$ and $n_j(\cdot)$, then even the shocks that affect $\chi^j(s^t)$ will leave u_c unaltered.

Result 3 concerning the inter-temporal MRS in consumption and Result 4 concerning efficient investment level satisfying EE are exactly the same as in the benchmark model, except for the fact that now the second argument in the consumers' utility function becomes $\chi^j(s^t) n_j(s^t)$ instead of $n_j(s^t)$.

- In case individuals differ in their discount rates, but these rates are time-invariant, we have the planner solving

$$\max_{\{c_j(\cdot), n_j(\cdot), K(\cdot)\}} \sum_{j=1}^{J} \psi_j \sum_{t,s^t} \beta_j^t \pi(s^t) u^j(c_j(s^t), n_j(s^t)) \tag{1.12}$$

[3] Higher $\chi^j(s^t)$ in the denominator reduces the fraction, and as $f_K(K,N)$ is decreasing in K, this means that $K^*(z^t)$ has to be reduced in response to an increase in $\chi^j(s^t)$, all other things equal.
[4] As before, the condition for $K(\cdot)$ does not change.

subject to the sequence of aggregate resource constraints as given by (1.2). Once again, the relevant FOC's are the ones for $c^j(\cdot)$ and $n^j(\cdot)$:

$$\psi_j \beta_j^t \pi(s^t) u_c^j(c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t)$$

$$-\psi_j \beta_j^t \pi(s^t) u_n^j(c_j^*(s^t), n_j^*(s^t)) = \mu_j \lambda(z^t) \pi(h^t) A(z^t) \theta^j(s^t) f_n(K^*(z^{t-1}), N^*(z^t))$$

As we divide the second equation by the first one, β_j cancels out and Result 1 goes through: regardless of the degree of impatience, efficiency requires for each household's MRS between consumption and leisure to be equal to the marginal product of labor. It is unsurprising, given the static nature of this condition.

Result 2 is affected by β_j's. Both household's marginal disutility from work and the marginal valuation for consumption varies with β_j's.

Result 3 is the affected: cross-sectional variation in the discount rates must be reflected in the households' inter-temporal MRS: for any j we have

$$\beta_j^t \pi(z^t) \frac{u_c^j(c_j^*(s^t), n_j^*(s^t))}{u_c^j(c_j^*(s^0), n_j^*(s^0))} = \frac{\lambda(z^t)}{\lambda(z^0)} \tag{1.13}$$

and so the two households types' i and j (with $i \neq j$) inter-temporal MRS must be related according to

$$\beta_j^t \frac{u_c^j(c_j^*(s^t), n_j^*(s^t))}{u_c^j(c_j^*(s^0), n_j^*(s^0))} = \beta_i^t \frac{u_c^i(c_i^*(s^t), n_i^*(s^t))}{u_c^i(c_i^*(s^0), n_i^*(s^0))} \tag{1.14}$$

Clearly, those who are more patient (have higher β_j) must have lower MRS, postponing consumption, i.e. having higher $c_j(s^t)$ and lower $c_j(s^0)$, all other things equal.

But idiosyncratic shocks h^t doesn't affect household's choice.

Result 4 stays intact. Notice, the original Euler Equation depends on $\lambda(z^t)$, which is independent of idiosyncratic terms.

In case if the discount rate is also *state-dependent*, we have the planner solving

$$\max_{\{c_j(\cdot), n_j(\cdot), K(\cdot)\}} \sum_{j=1}^{J} \psi_j \sum_{t, s^t} \pi(s^t) u^j(c_j(s^t), n_j(s^t)) \prod_{\tau=0}^{t} \beta_j(s^\tau) \tag{1.15}$$

with the FOC's for consumption and labor supply being

$$\psi_j \pi(s^t) u_c^j(c_j^*(s^t), n_j^*(s^t)) \prod_{\tau=0}^{t} \beta_j(s^\tau) = \mu_j \lambda(z^t) \pi(h^t)$$

$$-\psi_j \pi(s^t) u_n^j(c_j^*(s^t), n_j^*(s^t)) \prod_{\tau=0}^{t} \beta_j(s^\tau) = \mu_j \lambda(z^t) \pi(h^t) A(z^t) \theta^j(s^t) f_n(K^*(z^{t-1}), N^*(z^t))$$

Result 1 is unaffected.

This generally breaks down Result 2: household-specific shocks which affect not only $\theta^j(s^t)$ but also $\beta_j(s^t)$ *will* have an impact on the marginal utility of consumption, which can be clearly seen from the FOC for $c^j(\cdot)$.

Result 3 is affected as in the former paragraph.

Result 4 is unaffected.

EE for each household becomes

$$1 = \sum_{z^{t+1} \succ z^t} \beta_j(s^{t+1}) \pi(z^{t+1}|z^t) \frac{u_c^j(c_j^*(s^{t+1}), n_j^*(s^{t+1}))}{u_c^j(c_j^*(s^t), n_j^*(s^t))} \times$$

$$\left\{A(z^{t+1}) f_K(K^*(z^t), N^*(z^{t+1})) + 1 - \delta\right\}$$

But remember it's equivalent to

$$1 = \sum_{z^{t+1} \succ z^t} \frac{\lambda(z^{t+1})}{\lambda(z^t)} \left\{A(z^{t+1}) f_K(K^*(z^t), N^*(z^{t+1})) + 1 - \delta\right\}$$

- In case of the exogenous process for government spending $G(z^t)$, the only thing that is altered is the resource constraint, which now becomes

$$C(z^t) + K(z^t) + G(z^t) \leq A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta) K(z^{t-1}), \quad \forall z^t \tag{1.16}$$

  Since consumers may be taxed in a lump-sum fashion (one possible way for the government would be to make $T(z^t) = G(z^t)$ for each state and date), the only thing that changes due to $\{G(\cdot)\}$ is that the present value of aggregate consumption has to be reduced. Definitely, this would affect the shadow values of the aggregate resource constraints, captured by Lagrange multipliers $\lambda(z^t)$, but in other respects, the FOC's would go through without change, meaning that Results 1-4 continue to hold.

- Finally, when the investment adjustment cost $\Phi(I_t)$ is present, the planner's objective remains unchanged, while the aggregate resource constraint becomes

$$\Phi(K(z^t) - (1-\delta) K(z^{t-1})) + C(z^t) + K(z^t)$$

$$\leq A(z^t) f(K(z^{t-1}), N(z^t)) + (1-\delta) K(z^{t-1}), \quad \forall z^t \tag{1.17}$$

  where we have substituted for $I(z^t)$ from the law of motion for capital.

  In contrast to the previous items, here the FOC's with respect to $c^j(\cdot)$ and $n^j(\cdot)$ do not change, and thus Results 1-3 go through. On the other hand, the FOC with respect to $K(\cdot)$ now becomes

$$\begin{aligned} &\lambda(z^t) \left[1 + \Phi'\left(K^*(z^t) - (1-\delta) K^*(z^{t-1})\right)\right] = \\ &\sum_{z^{t+1} \succ z^t} \lambda(z^{t+1}) \left\{A(z^{t+1}) f_K(K^*(z^t), N^*(z^{t+1})) + \right. \\ &\left. (1-\delta) \left[1 + \Phi'\left(K^*(z^{t+1}) - (1-\delta) K^*(z^t)\right)\right]\right\} \end{aligned} \tag{1.18}$$

  So, now the inter-temporal Euler Equation has to be modified so as to incorporate the adjustment costs. However, the interpretation is the same: the left-hand side of (1.18) represents the marginal costs of an additional unit of capital, while the right-hand side is the marginal benefit from $K(z^t)$.

Remark. One important assumption making our life easier is that $\Phi(0) = \Phi'(0) = 0$: that is, small capital adjustments are essentially costless. For a model with fixed adjustment costs, see Caballero and Engel (1991).

2 Indivisible Labor

In this problem, we deal with the setting, where individual's labor supply is a zero/one decision: for any z^t, we have $n(z^t) \in \{0, 1\}$. Additionally, let us assume that there is a single type of household ($J = 1$), and that there is no *idiosyncratic* uncertainty, so that we can dispense with the s^t notation: in what follows, all allocations will be contingent on z^t only.

- Denote by $\phi(z^t)$ the measure of households whom the planner requires to work (i.e. choose $n(z^t) = 1$) in state z^t. The planner's problem is to choose the sequences of $\{c_0(\cdot), c_1(\cdot), \phi(\cdot), K(\cdot)\}$, solving

$$\max_{\{c_0(\cdot),c_1(\cdot),\phi(\cdot),K(\cdot)\}} \sum_{t,z_t} \beta^t \pi(z^t) \left[(1 - \phi(z^t))u(c_0(z^t), 0) + \phi(z^t)u(c_1(z^t), 1)\right] \tag{2.1}$$

subject to the aggregate resource constraint

$$C(z^t) + K(z^t) \leq A(z^t)f(K(z^{t-1}), N(z^t)) + (1 - \delta)K(z^{t-1}), \quad \forall z^t \tag{2.2}$$

where $C(z_t) = (1 - \phi(z^t))c_0(z^t) + \phi(z^t)c_1(z^t)$ and $N(z^t) = \phi(z^t)$.

The Lagrangian may be written as

$$\begin{aligned} \mathcal{L} =& \sum_{t,z_t} \beta^t \pi(z^t) \left[(1 - \phi(z^t))u(c_0(z^t), 0) + \phi(z^t)u(c_1(z^t), 1)\right] + \\ & \sum_{z^t} \lambda(z^t) \left[A(z^t)f(K(z^{t-1}), N(z^t)) + (1 - \delta)K(z^{t-1}) - K(z^t) - C(z^t)\right] \end{aligned} \tag{2.3}$$

and the FOC's with respect to $c_0(\cdot)$, $c_1(\cdot)$ and $\phi(\cdot)$ are, correspondingly:[5]

$$\beta^t \pi(z^t)[1 - \phi^*(z^t)]u_c(c_0^*(z^t), 0) = \lambda(z^t)[1 - \phi^*(z^t)]$$

$$\beta^t \pi(z^t)\phi^*(z^t)u_c(c_1^*(z^t), 1) = \lambda(z^t)\phi^*(z^t)$$

$$-\beta^t \pi(z^t)[u(c_1^*(z^t), 1) - u(c_0^*(z^t), 0)] = \lambda(z^t) \left[A(z^t)f_n(K^*(z^{t-1}), \phi^*(z^t)) - c_1^*(z^t) + c_0^*(z^t)\right]$$

- From the above FOC's on $c_0(\cdot)$ and $c_1(\cdot)$ we can see that optimality would require

$$u_c(c_0^*(z^t), 0) = u_c(c_1^*(z^t), 1) \tag{2.4}$$

When utility is separable in consumption and labor, i.e. $u(c, n) = v(c) - w(n)$, this condition reduces to

$$v'(c_0^*(z^t)) = v'(c_1^*(z^t)) \Longleftrightarrow c_0^*(z^t) = c_1^*(z^t) \tag{2.5}$$

That is to say, consumption should not depend on the employment status: optimality requires all the idiosyncratic risk to be diversified away.

[5] FOC for capital $K(\cdot)$ is exactly the same as in the benchmark case, so we do not reproduce it here.

- Recall that for each z^t, the "inner" planning problem is to allocate within period consumption and labor supply requirements $\{c_0, c_1, \phi\}$ to maximize the weighted sum of current period utilities:[6]

$$U(C, N; z^t) = \max_{c_0, c_1, \phi} \{(1-\phi)u(c_0, 0) + \phi u(c_1, 1)\} \tag{2.6}$$

subject to

$$C \geq (1-\phi)c_0 + \phi c_1 \tag{2.7}$$
$$N \leq \phi \tag{2.8}$$

Clearly, both constraints have to bind at the optimum, for otherwise the planner could increase the agent's utility by either marginally increasing c_0 and c_1 (in case (2.7) were slack), or else by marginally reducing ϕ (if (2.8) were slack).

Moreover, if $u(c, n) = v(c) - w(n)$, from the previous item we know that at the optimum, we would have $c_0^* = c_1^* = c^*$, which from (2.7) should be equal to C.

Planner's utility becomes

$$\begin{aligned} U(C, N; z^t) &= v(C) - (1-N)w(0) - Nw(1) \\ &= v(C) - [w(1) - w(0)]N - \text{constant} \end{aligned} \tag{2.9}$$

As we can see, the central planning problem maximizes the preferences of the representative agent, whose preferences are *linear*, and not (strictly) *concave*, in his labor supply N. In turn, this would imply a higher elasticity of labor supply. It is less costly (in terms of utility) for the representative individual to alter his n (at the aggregate). Thus, aggregate employment naturally gets to be more responsive to changes in the wage rate:

> ... [h]ence there is a discrepancy between the true preferences of agents and the preferences of the hypothetical representative consumer generating aggregate fluctuations. In particular, the second of these has preferences linear in n, indicating a higher elasticity of labor supply.[7]

- In the decentralized version of this economy, the representative consumer chooses the sequences of consumption, labor supply and asset positions, $\{c(\cdot), n(\cdot), a(\cdot)\}$, maximizing his expected present value of utility,

$$\max_{\{c(\cdot), n(\cdot), a(\cdot)\}} \sum_{t, z^t} \beta^t \pi(z^t) u(c(z^t), n(z^t)) \tag{2.10}$$

subject to $n(z^t) \in \{0, 1\}$ the following sequence of budget constraints:

$$c(z^t) + \sum_{z^{t+1} \succ z^t} \frac{q(z^{t+1})}{q(z^t)} a(z^{t+1}) \leq a(z^t) + W(z^t)n(z^t) + D(z^t) \tag{2.11}$$

where $q(z^t) = \lambda(z^t)/\lambda(z^0)$ is the price of an Arrow security,[8] $W(z^t)$ is the wage rate in the z^t state, given by

$$W(z^t) = A(z^t) f_n(K^*(z^{t-1}), N^*(z^t)) \tag{2.12}$$

[6] Dependence of the variables c_0, c_1 and ϕ on z^t in the objective has been suppressed for simplicity.
[7] Quoted from Rogerson (1988), p. 14.
[8] Recall that it denotes the price relative to z^0 of a security that pays 1 unit of the consumption good, if the aggregate state z^t realizes.

as derived from the firm's profit maximization, while $D(z^t)$ is the dividend paid by the representative firm:

$$D(z^t) = A(z^t)f(K(z^{t-1}), N(z^t)) + (1-\delta)K(z^{t-1}) - K(z^t) - W(z^t)N(z^t) \tag{2.13}$$

The first thing to note is that with a single type of household, the only sequence of asset positions consistent with market clearing is $a(z^t) \equiv 0$ for all t and z^t, since there is no individual risk to be diversified away through inter-temporal trade.

Given this observation, the consumer's state-z^t budget constraint simplifies to

$$c(z^t) \leq W(z^t)n(z^t) + D(z^t) \tag{2.14}$$

Now recall that $n(z^t)$ can be either 0 or 1. Hence, those who choose $n(z^t) = 1$ can afford to consume $W(z^t) + D(z^t)$ in state z^t, while those who choose $n(z^t) = 0$ can only consume $D(z^t)$. In turn, this implies that consumption – and hence, marginal utility of consumption – would differ across households, with those who choose $n(z^t) = 1$ having lower u_c than those who choose $n(z^t) = 0$.

This is different from the optimality condition (2.4) for the central planner's problem that we reached above. The reason that the Second Welfare Theorem fails is the inherent non-convexity of individuals' consumption sets due to the binary nature of n: we have $X = (c, n) \in \mathbb{R}_+ \times \{0, 1\}$.

On top of that, the planner's choice set *was* convex through the choice of $\phi(z^t)$, and thus the set of feasible allocations was larger than what is achievable by the competitive economy, rendering the First Welfare Theorem invalid as well:

> The solution to this apparent logical inconsistency is that the alternative allocation described above not belong to the set X and hence is outside the scope of the set of allocations considered by the first welfare theorem. Recall that the standard method of proof for the first welfare theorem involves an argument that if individuals prefer an allocation to their individual allocation then it must cost too much relative to their budget or else they would have purchased it. This argument does not hold here because the allocation involving lotteries is not viewed as a feasible one by consumers with consumption set X.[9]

- The representative household may now choose a lottery, which would require him to work with probability $\phi(z^t)$ when state z^t is realized, in exchange for the income $w(\phi(z^t), z^t)$ received.[10] Formally, the household will be solving

$$\max_{\{c_0(\cdot), c_1(\cdot), \phi(\cdot)\}} \sum_{t, z^t} \beta^t \pi(z^t) \left[(1 - \phi(z^t)) u(c_0(z^t), 0) + \phi(z^t) u(c_1(z^t), 1) \right] \tag{2.15}$$

subject to[11]

$$c_0(z^t) \leq w(\phi(z^t), z^t) + D(z^t) \tag{2.16}$$

$$c_1(z^t) \leq w(\phi(z^t), z^t) + D(z^t) \tag{2.17}$$

[9] See Rogerson (1988), p. 7.

[10] Conditional on z^t, this income $w(\phi, z^t)$ is going to be non-stochastic, i.e. the household is going to receive it regardless of the realization of the lottery (i.e. whether he works or not).

[11] Trade in Arrow securities will be redundant in the presence of lotteries, hence we simplify the budget constraint by eliminating the choice of wealth holdings $\{a(\cdot)\}$ from the consumer's problem.

This problem is very similar to the planner's problem (2.1)-(2.2) we dealt with at the beginning of this exercise. Clearly, the two budget constraints imply that consumption should be disentangled from the labor supply choice: $c_0^*(z^t) = c_1^*(z^t)$ for all z^t. The FOC on $\phi(\cdot)$ yields

$$-\beta^t \pi(z^t)\left[u(c_1^*(z^t),1) - u(c_0^*(z^t),0)\right] = \lambda(z^t) w_\phi(\phi^*(z^t), z^t) \tag{2.18}$$

where $\lambda(z^t)$ is the sum of Lagrange multipliers for the household's budget constraints (2.16) and (2.17). On the other hand, the representative firm chooses capital $K^*(z^t)$ and employment $N^*(z^t)$, solving

$$\max_{\{K(\cdot),N(\cdot)\}} \left\{A(z^t) f(K(z^{t-1}), N(z^t)) - w(z^t)N(z^t) - r(z^t)K(z^t)\right\} \tag{2.19}$$

where $w(z^t)$ and $r(z^t)$ are the wage rate and the rental price of capital in state z^t, respectively. The FOC with respect to $N(z^t)$ gives us[12]

$$A(z^t) f_n(K^*(z^{t-1}), N^*(z^t)) = w(z^t) \tag{2.20}$$

Now observe that since the Law of Large Numbers is assumed to hold here, $\phi(z^t)$ also equals the measure of those households who work in state z^t. If every working household receives wage $w(z^t)$ from the firm, the total wage bill is given by $w(\phi(z^t), z^t) = \phi(z^t)w(z^t)$, – and so, we have $w_\phi(\phi^*(z^t), z^t) = w(z^t)$.

Moreover, labor market clearing condition requires $N^*(z^t) = \phi^*(z^t)$ for each z^t. Using these results and combining (2.18) with (2.20), we eventually arrive at exactly the same condition we had before:

$$-\beta^t \pi(z^t)\left[u(c_1^*(z^t),1) - u(c_0^*(z^t),0)\right] = \lambda(z^t) A(z^t) f_n(K^*(z^{t-1}), \phi^*(z^t)) \tag{2.21}$$

The interpretation of (2.21) is quite natural: the left-hand side represents (expected) marginal cost of increasing $\phi(\cdot)$, captured by the utility difference $u(c_1^*(z^t),1) - u(c_0^*(z^t),0)$, while the right-hand side reflects the marginal benefit from raising $\phi(\cdot)$, captured by the marginal product of labor, $A(\cdot)f_n(\cdot)$.

To conclude, introducing lotteries into the household's problem convexifies the consumption set, rendering *both* the First and the Second Welfare Theorems applicable.

- One obvious advantage of the Rogerson's model is that it reconciles the fact that estimated labor-supply elasticities in micro studies appear to be much smaller than what are needed to generate aggregate fluctuations observed in the data. In the case studied here, the aggregate economy behaves as if there was no non-convexities but all individuals have preferences which are linear in labor even though no individual in the economy has such preferences. [13] To get a flavor of the magnitudes in question, keep in mind that about two-thirds of business cycle fluctuations are accounted for by the variation in labor supply (another third comes from the variation in TFP), and around three-quarters of the variation in the labor force occurs due to adjustment on the extensive margin, i.e. composed of movements in and out of the labor force.

[12] We are not interested in the FOC for $K(\cdot)$ at this point.

[13] Empirical studies conducted by labor economists usually report small covariation between hours worked and wages for prime-aged males. At the same time, macroeconomists report a much higher elasticity based on *aggregate hours* worked. See, for instance, Altonji and Ashenfelter (1980) or Kydland and Prescott (1982).

In addition, several other situations relevant for macroeconomic issues involve binary decision on the part of households: think of the real estate market, or the market for cars or consumer durables. Similar indivisibilities generating non-convex choice sets are present in marriage, fertility decisions, or occupational choice. Implications of these non-convexities for aggregate fluctuations can be analyzed within a model along the lines of Rogerson (1988).

However, this model is quite controversial in several respects. One major problem deals with the way to interpret these lotteries: What could be their counterpart in reality? Which labor-market institutions might resemble the stochastic choice of employment referred to above? Certainly, one might think of the government introducing labor-income taxes and providing unemployment insurance – these measures tend to bring closer the consumption of working and non-working individuals ($c_0(\cdot)$ and $c_1(\cdot)$ in the above notation). Yet in reality this consumption smoothing is far from perfect: empirical studies usually document a statistically significant drops in consumption in response to entering the unemployment pool.

Another potential drawback of this framework is that it allows too little flexibility in the employment: more often than not employment decisions are more complex than the binary 0/1 choice: to some extent the number of hours worked might be subject to choice (e.g. for the unskilled labor in the manufacturing sector); else there is usually some degree of flexibility in the career choice of those who obtain higher education.

Finally, Rogerson (1988) formulated the model in a static framework. It is an open question how easily the discrepancy between competitive equilibrium and the first best goes through in a dynamic setting. In particular, above we precluded trade in contingent claims on the grounds that all households are ex-ante identical, implying there are no gains from inter-temporal trade. Yet we conjecture that Arrow securities might replicate the lotteries described above.[14]

To conclude, there are still considerable disagreements among the economists concerning the proper magnitude of the labor supply elasticity. A good account of this issue, together with a quick review of up-to-date literature in the field, can be found in Prescott and Wallenius (2012).

3 Multiple Industries

Now we deal with an economy where there are K different industries. For this problem we would assume that there is a single type of household ($J = 1$).[15]

- The planner chooses $\{c_1(\cdot), \ldots, c_K(\cdot), n_1(\cdot), \ldots, n_K(\cdot), K_1(\cdot), \ldots, K_K(\cdot)\}$ solving

$$\max_{\{c,n,K\}} \sum_{t,z^t} \beta^t \pi(z^t) u(c_1(z^t), \ldots, c_K(z^t), n(z^t)) \tag{3.1}$$

[14] To fix ideas, think of an environment where there is no aggregate uncertainty, and for simplicity suppose that $\phi^* = 1/2$. Let $i \in [0, 1/2]$ households work in even periods and lend to those who do not work, while the other $i \in [1/2, 1]$ work and repay their debts in odd periods. Our guess is that for β sufficiently close to 1, this arrangement will be incentive compatible, i.e. no one would want to (unilaterally) deviate. This issue is subject to further research, and we invite those who are interested to discuss it together.

[15] The extension to $J > 1$ is straightforward, but cumbersome, and offers no additional insight.

Assuming that capital is industry-specific, i.e. that $K_k(\cdot)$ can be made only out of output k produced, the planner would face the following K resource constraints in each state z^t:

$$c_k(z^t) + K_k(z^t) \leq A_k(z^t) f(K_k(z^{t-1}), N_k(z^t)) + (1-\delta)K_k(z^{t-1}), \quad \forall z^t \tag{3.2}$$

for each $k = 1, \ldots, K$. The associated Lagrangian would be

$$\mathcal{L} = \sum_{t, z^t} \beta^t \pi(z^t) u(c_1(z^t), \ldots, c_K(z^t), n(z^t)) +$$

$$\sum_{z^t} \sum_{k=1}^{K} \lambda_k(z^t) \left[A_k(z^t) f(K_k(z^{t-1}), N_k(z^t)) + (1-\delta)K_k(z^{t-1}) - K_k(z^t) - c_k(z^t) \right]$$

Given that with the single type of household, for each z^t it must be the case that

$$n(z^t) = \sum_{k=1}^{K} n_k(z^t) = \sum_{k=1}^{K} N_k(z^t) \tag{3.3}$$

the FOC's with respect to $c_k(\cdot)$ and $N_k(\cdot)$ are, correspondingly:

$$\beta^t \pi(z^t) u_{c_k}(c_1^*(z^t), \ldots, c_K^*(z^t), n^*(z^t)) = \lambda_k(z^t)$$

$$-\beta^t \pi(z^t) u_n(c_1^*(z^t), \ldots, c_K^*(z^t), n^*(z^t)) = \lambda_k(z^t) A_k(z^t) f_n(K_k^*(z^{t-1}), N_k^*(z^t))$$

while the one for $K_k(\cdot)$ is

$$\lambda_k(z^t) = \sum_{z^{t+1} \succ z^t} \lambda_k(z^{t+1}) \left\{ A_k(z^{t+1}) f_K(K_k^*(z^t), N_k^*(z^{t+1})) + 1 - \delta \right\}$$

- In case in the cross-partial $u_{ij} = 0$ for $i \neq j$, the marginal utility from consumption of the i'th good, $u_{c_i}(c_1(z^t), \ldots, c_K(z^t), n(z^t))$, does not depend on consumption of other goods – hence $c_i(z^t)$ is completely pinned down by $\lambda_i(z^t)$: this implies that aggregate shocks hitting i'th industry (increasing the shadow value of the relevant resource constraint $\lambda_i(z^t)$) translate directly to a drop in consumption of the corresponding commodity, $c_i(z^t)$, but have no repercussive effects on other industries $j \neq i$.

- For each industry, $k = 1, \ldots, K$, the following condition has to be satisfied:

$$-\frac{u_n(c_1^*(z^t), \ldots, c_K^*(z^t), n^*(z^t))}{u_{c_k}(c_1^*(z^t), \ldots, c_K^*(z^t), n^*(z^t))} = A_k(z^t) f_n(K_k^*(z^{t-1}), N_k^*(z^t))$$

  This implies that the marginal product of labor times the marginal utility from consumption of good k have to be equalized across industries (and in turn, have to be equal to the marginal utility of leisure $-u_n(\cdot)$). Whenever a shock z^t results in the increased productivity of sector k, this translates into an increase in employment $N_k^*(z^t)$, together with a rise in consumption of good k, $c_k(z^t)$.

- The multi-industry extension of the basic framework is suitable to address the question of how and why the allocation of resources across different industrial sectors varied in the course of history. The above formulation implicitly postulates the variation in the rates of technical change, the $\{A_1(\cdot), \ldots, A_K(\cdot)\}$ sequence, to be the main driving force behind the observed reallocation. Yet several modifications might be employed in order to capture the competing hypotheses for the observed transition from agriculture to manufacturing to service. Changing tastes can be formalized via the vector of taste shocks $\{\chi_1(\cdot), \ldots, \chi_K(\cdot)\}$ entering multiplicatively into the utility function: $u(\chi_1(z^t)c_1(z^t), \ldots, \chi_K(z^t)c_K(z^t), n(z^t))$. In that case, the FOC for good-k consumption would become

$$\beta^t \pi(z^t)\chi_k(z^t)u_{c_k}(\chi_1(z^t)c_1(z^t), \ldots, \chi_K(z^t)c_K(z^t), n(z^t)) = \lambda_k(z^t) \tag{3.4}$$

In this respect, a sectoral shift, say, from manufacturing to services, can be accounted for not by the fact that the later became more productive in the course of time (via a lower Lagrange multiplier $\lambda(z^t)$ on the right-hand side), but simply because the valuation for services has increased (via a larger $\chi_k(z^t)$ value on the left-hand side).

The change in the importance of skills and capital in different sectors might be formalized by indexing the set of production functions by k, i.e. having a sequence of $\{f^k(K_k(\cdot), N_k(\cdot))\}$. However, this would offer too many degrees of freedom: to discipline the study somehow, one might stick to the single functional form f, but instead of having a single multiplicative shock $A_k(z^t)$, introduce separate shocks to labor and capital – that is, dealing with $f(A_k^K K(z^{t-1}), A_k^N N(z^t))$.

Finally, non-homothetic preferences can be allowed for by appropriate choice of the utility function. For instance, the necessity of food might be captured by specifying a minimum subsistence level in the food consumption, below which the household's utility becomes minus infinity. However, such modification is likely to render the model highly non-tractable,[16] and one would need to rely on numerical solution.

To test these alternative hypotheses, one would need to have data on the Solow residuals (for $\{A(z^t)\}$ processes), disaggregated into agricultural, manufacturing and service sectors, respectively. Additionally, estimates for returns on labor and capital in these industries would be needed to test for time-varying skill premia. Regarding the explanations based on change in tastes or non-homotheticities, one would need to look for micro data estimating different utility functions.

- Since in this model labor brings the same disutility regardless of where it is employed, and as there are no barriers to labor mobility, one would expect that quite naturally, over the cycle labor will move to the industry with the highest marginal product of labor.

4 Two Countries Open to Trade

This problem is a modified version of Backus et al. (1992). We consider the case where the social planner acts like the 'United Nations'. He is maximizing the joint welfare of the two countries subject to the resource constraints (probably with some allocated weights).

- This is just a standard question with two types of households, two industries producing homogenous products (both subject to decreasing returns to scale), two capitals. Households are immobile across the industries but capital can be allocated costlessly to any

[16] In particular, the complementarities u_{ij} will be non-zero.

country. The social planner's problem can be written as[17]

$$\max_{\{C(\cdot),N(\cdot),K(\cdot)\}} \sum_{t,z^t} \beta^t \pi(z^t) \left[U(C_1(z^t), N_1(z^t)) + U(C_2(z^t), N_2(z^t)) \right]$$

subject to

$$\sum_{i=1}^{2} \left[C_i(z^t) + K_i(z^t) - (1-\delta) K_i(z^{t-1}) \right] \leq \sum_{i=1}^{2} A_i(z^t) F_i(N_i(z^t), K_i(z^{t-1}))$$

The FOC's are:

$$\beta^t \pi(z^t) U_C(C_i(z^t), N_i(z^t)) = \lambda(z^t)$$

$$\beta^t \pi(z^t) U_N(C_i(z^t), N_i(z^t)) = -\lambda(z^t) A_i(z^t) F_{i,n}(N_i(z^t), K_i(z^{t-1}))$$

$$\lambda(z^t) = \sum_{z^{t+1} > z^t} \lambda(z^{t+1}) \left[A_i(z^t) F_{i,k}(N_i(z^t), K_i(z^{t-1})) + (1-\delta) \right]$$

From the FOC's,

$$\frac{U_C(C_1(z^t), N_1(z^t))}{U_C(C_2(z^t), N_2(z^t))} = 1$$

If the agents' utility functions are separable between consumption and leisure, then the marginal utility of consumption must equal in each period. We can clearly observe the perfect co-movement between consumption in the 2 countries.[18]

Let $A(t) = (A_1(t), A_2(t))^T$ be the vector of productivity, which involves according to the process:

$$A(t+1) = \Gamma A(t) + \varepsilon$$

where

$$\Gamma = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

and ε is a two-dimensional Gaussian vector with mean $(0,0)^T$ and variance-covariance matrix:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

ρ_{12} and ρ_{21} means the cross-section spill-over between the two countries. For simplicity, we assume that $\rho_{11} \gg \rho_{12}$, $\rho_{22} \gg \rho_{21}$ and $\rho_{11}, \rho_{22} > 0$.

As expected, when country 1 faces a bad productivity shock (or taste shock) relative to country 2, the relative amount of investment between the two countries increases, and the output ratio also increases. This indicates less co-movement between the countries' investment and output than consumption.

[17] We can add a weight ψ in front of the utility of country 1, which is the planner's weight. This will not alter the fundamentals of our analysis.

[18] If the utility is not separable, then the optimal allocation plan also depends on how labor choice affect the agent's marginal utility of consumption.

- We will integrate the two sub-questions together. The maximization problem can be written as:

$$\max_{C_H^1, C_F^1, C_H^2, C_F^1, N_1, N_2} \sum_{t, z^t} \beta^t \pi(z^t) \left[U(C_H^1, C_F^1, N_1) + U(C_H^2, C_F^2, N_2) \right]$$

s.t.

$$i = 1,2 \qquad C_H^i(z^t) + \tau C_F^j(z^t) + K_i(z^t) - (1-\delta) K_i(z^{t-1}) \leq A_i(z^t) F_i(N_i(z^t), K_i(z^{t-1}))$$

Note that in the equation above, we implicitly assumed that the capital is country specific : capital in country i can be made only out of good produced in country i.

Write down the Lagrangian and take the FOC, we get:

$$\beta^t \pi(z^t) U_{C_H}(C_H^i(z^t), C_F^i(z^t), N_i(z^t)) = \lambda_i(z^t)$$

$$\beta^t \pi(z^t) U_{C_F}(C_H^i(z^t), C_F^i(z^t), N_i(z^t)) = \tau \lambda_j(z^t)$$

$$\beta^t \pi(z^t) U_N(C_H^i(z^t), C_F^i(z^t), N_i(z^t)) = -\lambda_i(z^t) A_i(z^t) F_{i,n}(N_i(z^t), K_i(z^{t-1}))$$

$$\sum_{z^{t+1} > z^t} \lambda_i(z^{t+1}) \left[A_i(z^t) F_{i,k}(N_i(z^{t+1}), K_i(z^t)) + (1-\delta) \right] = \lambda_i(z^t)$$

Remember that there are three different elements comparing with the previous problem:

i. Home bias ($a > \frac{1}{2}$);

ii. Love for variety effect or imperfect substitution between goods ($\gamma < 1$);

iii. The shipping cost ($\tau > 1$).

Under the specification considered in this problem and letting

$$U_i(z^t) \equiv U(C_H^i(z^t), C_F^i(z^t), N_i(z^t))$$

we have

$$\left(\frac{U_j(z^t)}{U_i(z^t)} \right)^{1 - \frac{\gamma}{1-\theta}} \left(\frac{C_H^i}{C_F^j} \right)^{1-\gamma} = \frac{\tau a}{1-a} \tag{4.1}$$

Combining the two for $i = 1, 2$, we have:

$$\left(\frac{C_H^1 C_H^2}{C_F^1 C_F^2} \right)^{1-\gamma} = \left(\frac{\tau a}{1-a} \right)^2 \tag{4.2}$$

from which it can be seen that home good is more preferred at the world level if $\tau > 1$ and $a > \frac{1}{2}$. It is still true that C_H^i and C_F^j have positive co-movement, but this co-movement is diminished by a and τ.

- The model can be calibrated in the following way: parameters θ, γ, a and τ can be taken from empirical studies that estimated them, or else from calibrations of the closed-economy versions. The process for $A(\cdot)$ may be recovered from the data on Solow residuals.

Then one may approximate the model economy around the steady state (which can be computed analytically) and look at the implied means and standard deviations of the key variables: consumption, output, net exports (or their ratio to output), investment and saving. The corresponding counterparts in the data are readily available. One can further conduct a sensitivity analysis to see what appears to be the main driving force generating these magnitudes.

Finally, we can look at impulse responses to a one-time productivity shock in the home country of the key variables: C_t^j, Y_t^j, NX_t^j, I_t^j and S_t^j, for $j = h, f$. See Backus et al. (1992), pp. 756-768.

References

Altonji, J. and O. Ashenfelter (1980): "Wage Movements and the Labour Market Equilibrium Hypothesis," *Economica*, 47, 217–245.

Backus, D. K., P. J. Kehoe, and F. E. Kydland (1992): "International Real Business Cycles," *Journal of Political Economy*, 100, 745–775.

Caballero, R. J. and E. M. R. A. Engel (1991): "Dynamic (S, s) Economies," *Econometrica*, 59, 1659–1686.

Kydland, F. E. and E. C. Prescott (1982): "Time to Build and Aggregate Fluctuations," *Econometrica*, 50, 1345–1370.

Prescott, E. C. and J. Wallenius (2012): "Aggregate Labor Supply," *Federal Reserve Bank of Minneapolis Quarterly Review*, 35, 2–16.

Rogerson, R. (1988): "Indivisible Labor, Lotteries and Equilibrium," *Journal of Monetary Economics*, 21, 3–16.