Macroeconomics I—Problem Set 7: Applications of the planning problem Solutions

Christian Hellwig

TAs 2013: George Lukyanov, Harry Di Pei

TAs 2014: Constance de Soyres and Francois de Soyres

TAs 2015: Yue Fei and Lan Lan

November 27, 2015

1 RBC Model with GHH Preferences

This exercise familiarizes you with the model by Greenwood et al. (1988). In the neoclassical RBC framework augmented with the variable capacity utilization, they have shown how the shocks to the marginal productivity of capital propagate through the economy.

a. Since the production function $F(k_t h_t, l_t)$ is assumed to CRS, we have

$$F(\lambda k_t h_t, \lambda lt) = \lambda F(k_t h_t, l_t), \qquad \text{for any } \lambda > 0 \tag{1.1}$$

That is, the production function is *homogeneous of degree* 1. Then,[1] the function $F(\cdot)$ can be written as

$$F(k_t h_t, l_t) = k_t h_t F_1 + l_t F_2 \tag{1.2}$$

where F_i denotes the partial derivative with respect to the ith argument.[2] Differentiating (1.2) with respect to $k_t h_t$, we get

$$F_1 = F_1 + k_t h_t F_{11} + l_t F_{12} \quad \Longleftrightarrow \quad \frac{k_t h_t}{l_t} F_{11} + F_{12} = 0$$

while differentiation of (1.2) with respect to l_t yields

$$F_2 = k_t h_t F_{12} + F_2 + l_t F_{22} \quad \Longleftrightarrow \quad \frac{k_t h_t}{l_t} F_{12} + F_{22} = 0$$

Combining these two equations by substituting $\frac{k_t h_t}{l_t}$ term from one to the other, we get

$$-\frac{F_{12}}{F_{11}} \cdot F_{12} + F_{22} = 0 \quad \Longleftrightarrow \quad F_{11}F_{22} - F_{12}^2 = 0$$

what was to be shown.

[1] The idea behind the proof is to differentiate both sides of (1.1) with respect to λ and evaluate at $\lambda = 1$. To go further, you can have a look at a proof of Euler's Theorem, see e.g. Simon and Blume (1994), Theorem 20.4 on p. 491, or Intriligator (2002), p. 467.

[2] That is, $F_1 = \frac{\partial F(k_t h_t, l_i)}{\partial (k_t h_t)}$ and $F_2 = \frac{\partial F(k_t h_t, l_i)}{\partial (l_t)}$. We adopt a similar notation for $U(\cdot)$, $G(\cdot)$ and $\delta(\cdot)$ throughout.

b. Since the utility function takes the form

$$u(c_t, l_t) = U\left(c_t - G(l_t)\right) \tag{1.3}$$

taking the derivatives with respect to c_t and l_t and using the definition of MRS, we have

$$\text{MRS} \equiv -\frac{u_2(c_t, l_t)}{u_1(c_t, l_t)} = -\frac{U'\left(c_t - G(l_t)\right) \cdot [-G'(l_t)]}{U'\left(c_t - G(l_t)\right)} = G'(l_t) \tag{1.4}$$

As we can see, with these preferences, employment choice does not exhibit wealth effects, and hence, consumer's labor supply decision is disentangled from the intertemporal consumption-savings decision.

c. The representative household's problem is to choose the sequences of consumption, future-period capital, capacity utilization and labor or supply, $\{c_t, k_{t+1}, h_t, l_t\}$, in order to solve

$$\max_{\{c_t, k_{t+1}, h_t, l_t\}} \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \right\} \tag{1.5}$$

subject to the law of motion for the capital stock,

$$k_{t+1} = k_t \left[1 - \delta(h_t)\right] + i_t(1 + \varepsilon_t) \tag{1.6}$$

and the aggregate resource constraint

$$y_t = c_t + i_t \tag{1.7}$$

where $y_t = F(k_t h_t, l_t)$ and $\beta \in (0, 1)$.

Denoting the value function by $V(k_t; \varepsilon_t)$, we can write down recursively the following dynamic-programming problem:[3]

$$V(k_t; \varepsilon_t) = \max_{\{c_t, k_{t+1}, h_t, l_t\}} \left\{ U\left(c_t - G(l_t)\right) + \beta \int_Q V(k_{t+1}; \varepsilon_{t+1}) d\Phi(\varepsilon_{t+1} | \varepsilon_t) \right\} \tag{1.8}$$

subject to

$$c_t = F(k_t h_t, l_t) - \frac{k_{t+1}}{1 + \varepsilon_t} + \frac{k_t}{1 + \varepsilon_t} \cdot [1 - \delta(h_t)] \tag{1.9}$$

where $Q = [\underline{\varepsilon}, \overline{\varepsilon}]$ is the common support for $\{\varepsilon_t\}$, $\Phi(\cdot|\cdot)$ is the Markov transition function (denoting conditional cdf of ε_{t+1} given ε_t), and the condition (1.9) combines together the law of motion for capital (1.6) and the resource constraint (1.7). Substituting (1.9) into (1.8) for c_t and taking FOC's with respect to k_{t+1}, h_t and l_t, we get

$$\{k_{t+1}\}: \qquad \frac{U'\left(c_t - G(l_t)\right)}{1 + \varepsilon_t} = \beta \int_Q V_1(k_{t+1}; \varepsilon_{t+1}) d\Phi(\varepsilon_{t+1} | \varepsilon_t) \tag{1.10}$$

$$\{h_t\}: \qquad F_1(k_t h_t, l_t) = \frac{\delta'(h_t)}{1 + \varepsilon_t} \tag{1.11}$$

$$\{l_t\}: \qquad F_2(k_t h_t, l_t) = G'(l_t) \tag{1.12}$$

[3]The function V is thus representative agent's highest attainable utility when the current state is (k_t, ε_t).

Using the Envelope condition:

$$V_1(k_t;\varepsilon_t) = U'\left(c_t - G(l_t)\right)\left[F_1(k_t h_t, l_t)\cdot h_t + \frac{1-\delta(h_t)}{1+\varepsilon_t}\right] \tag{1.13}$$

we can rewrite (1.10) as

$$\frac{U'\left(c_t - G(l_t)\right)}{1+\varepsilon_t} = \beta\int_Q U'\left(c_{t+1} - G(l_{t+1})\right) \times \left[F_1(k_{t+1}h_{t+1}, l_{t+1})h_{t+1} + \frac{1-\delta(h_{t+1})}{1+\varepsilon_{t+1}}\right] d\Phi(\varepsilon_{t+1}|\varepsilon_t) \tag{1.14}$$

Equation (1.14) is the optimality condition governing investment: the left-hand side represents the current utility loss due to an extra unit of consumption foregone for investment, while the right-hand side is the discounted expected future utility obtained from an extra unit of investment made today. Condition (1.11) characterizes efficient capital utilization: it states that the capital should be utilized at a rate that equates marginal benefit of capital services to the marginal user cost.[4] Finally, (1.12) sets the marginal product of labor equal to the marginal disutility of working, measured in consumption units.

d. Notice that capital utilization, h_t, and labor supply, l_t, are jointly determined by conditions (1.11) and (1.12), whereas (1.14) pins down k_{t+1}, and hence c_t. In order to analyze the impact of the shock to the marginal efficiency of capital, ε_t, on h_t and l_t, let us implicitly differentiate the system (1.11)-(1.12) to get

$$k_t F_{11}(k_t h_t, l_t)\frac{dh_t}{d\varepsilon_t} + F_{12}(k_t h_t, l_t)\frac{dl_t}{d\varepsilon_t} - \frac{\delta''(h_t)}{1+\varepsilon_t}\frac{dh_t}{d\varepsilon_t} + \frac{\delta'(h_t)}{(1+\varepsilon_t)^2} = 0$$

$$k_t F_{12}(k_t h_t, l_t)\frac{dh_t}{d\varepsilon_t} + \left[F_{22}(k_t h_t, l_t) - G''(l_t)\right]\frac{dl_t}{d\varepsilon_t} = 0$$

or in the matrix form[5]

$$\begin{bmatrix} k_t F_{11} - \frac{\delta''}{1+\varepsilon_t} & F_{12} \\ k_t F_{12} & F_{22} - G'' \end{bmatrix} \times \begin{bmatrix} \frac{dh_t}{d\varepsilon_t} \\ \frac{dl_t}{d\varepsilon_t} \end{bmatrix} = \begin{bmatrix} -\frac{\delta'}{(1+\varepsilon_t)^2} \\ 0 \end{bmatrix}$$

Applying Cramer's rule, we get

$$\frac{dh_t}{d\varepsilon_t} = -\frac{\delta'(h_t)}{(1+\varepsilon_t)^2}\left[F_{22}(k_t h_t, l_t) - G''(l_t)\right] \times \Omega^{-1} \tag{1.15}$$

$$\frac{dl_t}{d\varepsilon_t} = \frac{\delta'(h_t)}{(1+\varepsilon_t)^2} k_t F_{12}(k_t h_t, l_t) \times \Omega^{-1} \tag{1.16}$$

[4] The latter is made up of two components: $\delta'(h_t)$ represents the cost of increased current depreciation from utilizing capital at a higher rate h_t, whereas $1/(1+\varepsilon_t)$ is the replacement cost of old in terms of new capital.

[5] To make things less cumbersome, we have suppressed the arguments.

where[6]

$$\Omega = \left[k_t F_{11} - \frac{\delta''}{1+\varepsilon_t}\right] \cdot \left(F_{22} - G''\right) - k_t F_{12}^2$$

$$= \left[k_t F_{11} - \frac{\delta''}{1+\varepsilon_t}\right] \cdot \left(F_{22} - G''\right) - k_t F_{11} F_{22}$$

$$= -k_t F_{11} G'' - \frac{\delta''}{1+\varepsilon_t} \cdot \left(F_{22} - G''\right) > 0$$

is the determinant. As can be clearly seen from (1.15) and (1.16), both the capacity utilization and the labor supply increase with ε_t.

The interpretation is as follows: an increase in the marginal efficiency of capital (ε_t) reduces the cost of capital utilization – the right-hand side of (1.11), – and hence induces a rise in h_t. Since $F_{12} > 0$ (the inputs are *complementary*), an increase in h_t (and thus in $k_t h_t$, since k_t is fixed) raises the marginal product of labor, resulting in higher l_t. Given that k_t is predetermined, while both h_t and l_t rise in response to an increase in ε_t, this immediately implies that the output y_t rises as well.

e. In order to analyze the impact of an increase in ε_t on k_{t+1} and c_t, let us implicitly differentiate the system given by (1.9) and (1.10). We have[7]

$$\frac{dc_t}{d\varepsilon_t} + \frac{1}{1+\varepsilon_t} \cdot \frac{dk_{t+1}}{d\varepsilon_t} - \left[\left(k_1 F_1 + \frac{\delta'}{1+\varepsilon_t}\right) \cdot \frac{dh_t}{d\varepsilon_t} + F_2 \cdot \frac{dl_t}{d\varepsilon_t}\right] - \frac{k_{t+1} - (1-\delta)k_t}{(1+\varepsilon_t)^2} = 0$$

$$-\frac{U'}{(1+\varepsilon_t)^2} + \frac{U''}{1+\varepsilon_t} \cdot \left[\frac{dc_t}{d\varepsilon_t} - G' \cdot \frac{dl_t}{d\varepsilon_t}\right] - \beta \int_Q V_{11} d\Phi \cdot \frac{dk_{t+1}}{d\varepsilon_t} = 0$$

or in the matrix form

$$\begin{bmatrix} \frac{1}{1+\varepsilon_t} & 1 \\ -\beta \int_Q V_{11} d\Phi & \frac{U''}{1+\varepsilon_t} \end{bmatrix} \times \begin{bmatrix} \frac{dk_{t+1}}{d\varepsilon_t} \\ \frac{dc_t}{d\varepsilon_t} \end{bmatrix} = \begin{bmatrix} \left(k_1 F_1 + \frac{\delta'}{1+\varepsilon_t}\right) \cdot \frac{dh_t}{d\varepsilon_t} + F_2 \cdot \frac{dl_t}{d\varepsilon_t} + \frac{k_{t+1}-(1-\delta)k_t}{(1+\varepsilon_t)^2} \\ \frac{U'}{(1+\varepsilon_t)^2} + G' \frac{U''}{1+\varepsilon_t} \frac{dl_t}{d\varepsilon_t} \end{bmatrix}$$

Plugging in the expressions for $dh_t/d\varepsilon_t$ and $dl_t/d\varepsilon_t$ derived in (1.15) and (1.16) above, and using the definition of investment: $i_t \equiv k_{t+1} - (1-\delta)k_t$, we eventually get[8]

$$\frac{dk_{t+1}}{d\varepsilon_t} = -\frac{U'}{U'' + \beta(1+\varepsilon_t)^2 \int_Q V_{11} d\Phi} + i_t \frac{U''}{U'' + \beta(1+\varepsilon_t)^2 \int_Q V_{11} d\Phi} > 0 \tag{1.17}$$

$$\frac{dc_t}{d\varepsilon_t} = \frac{F_2 F_{12} \delta'}{(1+\varepsilon_t)^2 \Omega} \cdot k_t + \frac{U'/(1+\varepsilon_t) + i_t \beta(1+\varepsilon_t) \int_Q V_{11} d\Phi}{U'' + \beta(1+\varepsilon_t)^2 \int_Q V_{11} d\Phi} \lesseqgtr 0 \tag{1.18}$$

[6] In the computation of Ω, we are using the fact that $F_{12}^2 = F_{11} F_{22}$, which we derived in (a) above.

[7] Recall that from now on, we assume that the shocks $\{\varepsilon_t\}$ are i.i.d.

[8] By applying standard reasoning in dynamic programming (see e.g. Stokey et al. (1989)), it can be shown that the value function $V(k_t; \varepsilon_t)$ is continuously differentiable, increasing and concave in its first argument. However, for (1.17) and (1.18) to be valid, we actually need to assume that $V(\cdot)$ is *twice* continuously differentiable.

As can be seen, ε_t has two effects on the capital stock in period $t+1$. The first term in (1.17) is a positive *substitution effect* due to increased productivity of the newly produced capital stock. The second term is the *income effect*: a given desired level of next period's capital stock can now be obtained with a lower level of current investment; part of these savings in current resource utilization goes to the increase in future capital stock.

The impact of a rise in ε_t on current consumption is ambiguous. It is affected in three distinct ways by the shock to ε_t. The term $[U'/(1+\varepsilon_t)]/[U''+\beta(1+\varepsilon_t)^2\int_Q V_{11}d\Phi] < 0$ in (1.18) illustrates the *intertemporal substitution effect* due to increase in productivity of the newly produced capital: the rise in rate of return dissuades consumption and promotes capital accumulation. The term $[i_t\beta(1+\varepsilon_t)\int_Q V_{11}d\Phi]/[U''+\beta(1+\varepsilon_t)^2\int_Q V_{11}d\Phi] > 0$ stands for the *income effect*, tending to increase current consumption. Finally, the term $[F_2F_{12}\delta']/[(1+\varepsilon_t)^2\Omega]\cdot k_t > 0$ represents the *intratemporal margin* between consumption and leisure: a higher utilization rate raises the marginal productivity of labor, generating a substitution effect away from leisure and toward consumption. Hence, there is a channel through which both consumption and investment can move proycically.

f. Implicit differentiation of the system (1.11)-(1.12) updated one period with respect to k_{t+1} (and fixing ε_{t+1}) yields

$$\left[k_{t+1}F_{11}-\frac{\delta''}{1+\varepsilon_{t+1}}\right]\frac{dh_{t+1}}{dk_{t+1}}+F_{12}\frac{dl_{t+1}}{dk_{t+1}}+h_{t+1}F_{11}=0$$

$$k_{t+1}F_{12}\frac{dh_{t+1}}{dk_{t+1}}+\left[F_{22}-G''\right]\frac{dl_{t+1}}{dk_{t+1}}+h_{t+1}F_{12}=0$$

or in the matrix form

$$\begin{bmatrix}\left(k_{t+1}F_{11}-\frac{\delta''}{1+\varepsilon_{t+1}}\right) & F_{12}\\ k_{t+1}F_{12} & F_{22}-G''\end{bmatrix}\times\begin{bmatrix}\frac{dh_{t+1}}{dk_{t+1}}\\ \frac{dl_{t+1}}{dk_{t+1}}\end{bmatrix}=\begin{bmatrix}-h_{t+1}F_{11}\\ -h_{t+1}F_{12}\end{bmatrix}$$

from which we get

$$\frac{dh_{t+1}}{dk_{t+1}}=\frac{F_{12}^2-F_{11}\left(F_{22}-G''\right)}{\Omega}\cdot h_{t+1}=\frac{F_{11}G''}{\Omega}h_{t+1}<0 \tag{1.19}$$

$$\frac{dl_{t+1}}{dk_{t+1}}=\frac{\delta''F_{12}}{(1+\varepsilon_{t+1})\Omega}\cdot h_{t+1}>0 \tag{1.20}$$

where

$$\Omega=\left[k_{t+1}F_{11}-\frac{\delta''}{1+\varepsilon_{t+1}}\right]\left(F_{22}-G''\right)-k_{t+1}F_{12}^2>0$$

and we once again made use of the fact that $F_{12}^2=F_{11}F_{22}$. So, we see that the optimal rate of utilization declines, since higher k_{t+1} reduces the marginal productivity of capital services. On the other hand, l_{t+1} increases, because the factors are complementary: higher k_{t+1} increases the marginal productivity of labor.

Finally, implicitly differentiating (1.14) forwarded one period and using (1.19)-(1.20), we eventually arrive at

$$\frac{dk_{t+2}}{dk_{t+1}}=\frac{U''\left[(1+\varepsilon_{t+1})F_1h_{t+1}+(1-\delta)\right]}{U''+\beta(1+\varepsilon_{t+1})^2\int_Q V_{11}d\Phi}>0 \tag{1.21}$$

g. In the standard RBC model along the lines of Kydland and Prescott (1982) or Long and Plosser (1983), capacity utilization was held fixed and shocks affected overall productivity. There, positive TFP shock had an impact on current consumption through the intertemporal substitution effect on leisure, whereas the resulting capital accumulation provided a channel of persistence (even in the case of serially uncorrelated technology shocks).

   In contrast, the model by Greenwood et al. (1988) deals with the shocks that affect efficiency of newly produced investment goods, which in turn induces more intensive utilization and accelerated depreciation of already installed capital. It could be shown that in the model with fixed capacity utilization, a shock to the productivity of capital would induce individuals not only to postpone leisure, but also to cut consumption, hence generating *negative* co-movement between consumption and output, and countercyclical labor productivity, contradicting empirical evidence. See Barro and King (1984) and the Appendix to Greenwood et al. (1988).

2 A Periodic Economy

This problem is taken from Ljungqvist and Sargent (2004); see Exercise 8.5 on p. 254.

a. The history s^t can be written as $s^t = (s_0, s_1, \dots, s_{t-1}, s_t)$. It has probability

$$\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_0)\pi(s_0) \tag{2.1}$$

   where $\pi(s'|s)$ is the conditional probability of transition from state s to state s'.

b. The planning problem is given by

$$\max_{c^1,c^2} \left\{\theta \ln\left(c^1\right) + (1-\theta)\ln\left(c^2\right)\right\} \tag{2.2}$$

   subject to

$$c^1(s^t) + c^2(s^t) \le Y(s^t), \quad \forall s^t \tag{2.3}$$

   where $\theta \in (0,1)$ is a Pareto weight on household 1 and $Y(s^t) = y^1(s^t) + y^2(s^t)$ denotes an aggregate endowment in state s^t. Substituting for $c^2(s^t)$ from (2.3) into (2.2), we have

$$\max_{c^1} \left\{\theta \ln\left(c^1\right) + (1-\theta)\ln\left(Y(s^t) - c^1\right)\right\}$$

   Taking FOC with respect to c^1 yields

$$\frac{\theta}{c^1} - \frac{1-\theta}{Y(s^t) - c^1} = 0 \quad \Longleftrightarrow \quad c^1(s^t) = \theta Y(s^t)$$

   which from the resource constraint implies

$$c^2(s^t) = (1-\theta)Y(s^t)$$

   Unsurprisingly, the higher is planner's weight on agent 1's utility, the higher is his consumption $c^1(s^t)$. Also observe that the ratio of consumptions is constant across states:

$$\frac{c^1(s^t)}{c^2(s^t)} = \frac{\theta}{1-\theta} \tag{2.4}$$

c. A competitive equilibrium is a feasible allocation of consumption and holdings of Arrow securities, $\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}$ for each agent $i = 1, 2$ and a pricing kernel $\{Q_t\}_{t=0}^{\infty}$, such that

   i) Given the pricing kernel, the allocation solves the household's problem, for all $i = 1, 2$.

   ii) Market-clearing conditions are satisfied: for all $t = 0, 1, \ldots$

$$\sum_{i=1}^{2} c_t^i = \sum_{i=1}^{2} y_t^i \quad \text{and} \quad \sum_{i=1}^{2} a_{t+1}^i = 0$$

d. Given any realization of the initial state s_0, individual i chooses the sequence of consumption and asset holdings, $\{c^i(s^t|s_0), a^i(s^{t+1}|s_0)\}_{t=0}^{\infty}$ in order to solve

$$\max_{\{c^i(s^t), a^i(s^{t+1})\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c(s^t|s_0)) \tag{2.5}$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} Q(s^t) c^i(s^t|s_0) \leq \sum_{t=0}^{\infty} \sum_{s^t} Q(s^t) y^i(s^t) \tag{2.6}$$

and $a^i(s_0) = 0$. Setting up the Lagrangian, we get

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) U(c(s^t|s_0)) + \mu_i \left\{ \sum_{t=0}^{\infty} \sum_{s^t} Q(s^t) \left[y^i(s^t) - c^i(s^t|s_0) \right] \right\} \tag{2.7}$$

Taking FOCs with respect to $c^i(s_0)$ and $c^i(s^t|s_0)$, we have

$$\pi(s_0) u'(c^i(s_0)) = \mu_i Q(s_0)$$

$$\beta^t \pi(s^t) u'(c^i(s^t|s_0)) = \mu_i Q(s^t)$$

Dividing one condition by the other, denoting $\pi(s^t|s_0) \equiv \frac{\pi(s^t)}{\pi(s_0)}$ and $Q(s^t|s_0) \equiv \frac{Q(s^t)}{Q(s_0)}$ and using the fact that $u(c^i(s^t)) = \ln\left(c^i(s^t)\right)$, we have

$$\beta^t \pi(s^t|s_0) \frac{c_0^1(s_0)}{c_t^1(s^t|s_0)} = Q(s^t|s_0) = \beta^t \pi(s^t|s_0) \frac{c_0^2(s_0)}{c_t^2(s^t|s_0)}$$

which in turn implies

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{c_0^2(s_0)}{c_0^1(s_0)}$$

Now, the market-clearing condition at time 0 implies

$$c_0^1(s_0) + c_0^2(s_0) = Y(s_0)$$

and so

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{Y(s_0) - c_0^1(s_0)}{c_0^1(s_0)}$$

From (b) we know that in the competitive equilibrium, $c(s_0)$ has to be equal to some fraction ξ of $Y(s_0)$. This implies

$$\frac{c_t^2(s^t|s_0)}{c_t^1(s^t|s_0)} = \frac{1-\xi}{\xi}$$

Finally, market-clearing at time t should satisfy

$$c_t^1(s^t|s_0) + c_t^2(s^t|s_0) = Y(s^t)$$

and therefore, we have

$$c_t^1(s^t|s_0) = \xi Y(s^t) \quad \text{and} \quad c_t^2(s^t|s_0) = (1-\xi)Y(s^t) \tag{2.8}$$

implying that prices of Arrow securities would be proportional to the time-0 aggregate endowment and inversely related to the aggregate endowment at date t:

$$Q(s^t|s_0) = \beta^t \pi(s^t|s_0)\frac{Y(s_0)}{Y(s^t)} \tag{2.9}$$

e. We have first computed the Pareto optimal allocation and then the competitive equilibrium. The two welfare theorems state conditions under which the Pareto optimal allocation can be implemented as a competitive equilibrium supported by prices $Q(s^t|s_0)$ and vice versa. The planner's weight θ on agent 1 has its counterpart in ξ.

f. A recursive competitive equilibrium is an initial wealth distribution $\{a^i(s_0)\}_{i=1}^2$, decision rules $\{c^i(a,s), a'^i(\theta,s')_{s'>s}\}_{i=1}^2$, a pricing kernel $Q(s'|s)$ and a pair of value functions $\{v^i(\theta,s)\}_{i=1}^2$ such that

   i) Given the pricing kernel and the initial wealth distribution, the decision rules solve each household's problem.

   ii) For all realizations $\{s_t\}_{t=0}^\infty$, the allocations implied by the decision rules satisfy market-clearing conditions:

$$\sum_{i=1}^2 c_t^i = \sum_{i=1}^2 y^i(s_t) \quad \text{and} \quad \sum_{i=1}^2 a_{t+1}^i(s') = 0$$

The natural borrowing limit rules out Ponzi schemes by restricting short positions in Arrow securities to be less than the present discounted value of all future income in each state tomorrow: for all $i = 1, 2$

$$-a_{t+1}^i(s') \leq \bar{A}^i(s') \tag{2.10}$$

where

$$\bar{A}^i(s') = \sum_{\tau \geq t} \sum_{s^\tau | s^t} Q(s^\tau|s^t) y^i(s^\tau|s^t)$$

We can recursively compute this this natural debt limit as

$$\bar{A}^i(s) = y^i(s) + \beta \sum_{s'>s} \pi(s'|s)\frac{Y(s)}{Y(s')}\bar{A}^i(s') \tag{2.11}$$

3 Preliminaries on Epstein and Zin Preference

a. Utility today:

$$V = F(c, G^{-1}\left[\mathbb{E}_t G(V_{t+1})\right]) = F(c, G^{-1}\left[0.5G(V_L) + 0.5G(V_H)\right])$$

The definition of a certainty equivalent under $G(,)$ is

$$G(\hat{V}) = 0.5G(V_L) + 0.5G(V_H)$$
$$\hat{V} = G^{-1}\left[0.5G(V_L) + 0.5G(V_H)\right]$$

So $R_t(V_{t+1}) = G^{-1}\left[\mathbb{E}_t G(V_{t+1})\right]$ is a certainty equivalent.

b.

$$V_t = \left\{(1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_t(V_{t+1}^{1-\alpha})\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}, \quad t = 1, \ldots, T \tag{3.1}$$

c. When $\rho \to 1$, $F(c,z) = c^{1-\beta}z^{\beta}$. When $\alpha \to 1$, $G(x) = log(x)$. $R_t(V_{t+1}) = E_t(V_{t+1}^{1-\alpha})^{\frac{1}{1-\alpha}}$ when $\alpha \in [0,1]$ and $R_t(V_{t+1}) = exp(E_t log(V_{t+1}))$ when $\alpha \to 1$

d. If no uncertainty, $R_t(V_{t+1}) = V_{t+1}$ and

$$V_t = \left\{(1-\beta)c_t^{1-\rho} + \beta V_{t+1}^{1-\rho}\right\}^{\frac{1}{1-\rho}} \tag{3.2}$$

Define

$$W_t = \frac{V_t^{1-\rho}}{(1-\beta)(1-\rho)}$$

(3.2) is equivalent to

$$W_t = \frac{c_t^{1-\rho}}{1-\rho} + \beta W_{t+1}$$

$$W_t = (1-\rho)c_t^{1-\rho} + \beta\Sigma_{s=0}^{\infty}\beta^s\frac{c_{t+s+1}^{1-\rho}}{1-\rho}$$

$$W_t = \Sigma_{s=0}^{\infty}\beta^s\frac{c_{t+s}^{1-\rho}}{1-\rho}$$

We recover the standard time-separable preferences with CRRA utility.

e. If $\alpha = \rho$,

$$V_t = \left\{(1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_t(V_{t+1}^{1-\alpha})\right]^{\frac{1-\rho}{1-\alpha}}\right\}^{\frac{1}{1-\rho}}$$

simplifies to

$$V_t^{1-\rho} = (1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_t(V_{t+1}^{1-\alpha})\right]$$
$$V_t^{1-\rho} = (1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_t(V_{t+1}^{1-\rho})\right]$$

As in question (4), define $W_t = \frac{V_t^{1-\rho}}{(1-\beta)(1-\rho)}$, we could get

$$W_t = \Sigma_{s=0}^{\infty}\beta^s\mathbb{E}_t(\frac{c_{t+s}^{1-\rho}}{1-\rho})$$

We recover the standard time-separable *expected* discounted utility with CRRA utility.

f. With standard time-separable expected discounted utility, we are indifferent between these lotteries.

Proof:

We compare the time 0 utility V_0 With standard time-separable expected discounted utility,

$$V_0 = E_0 \Sigma_{t=0}^{\infty} \beta^t \frac{c_t^{1-\rho}}{1-\rho}$$

For Lottery A,

$$V_{0,A} = \Sigma_{t=0}^{\infty} \beta^t (\frac{1}{2} \frac{c_h^{1-\rho}}{1-\rho} + \frac{1}{2} \frac{c_l^{1-\rho}}{1-\rho})$$

For Lottery B,

$$\begin{aligned} V_{0,B} &= \frac{1}{2} \Sigma_{t=0}^{\infty} \beta^t \frac{c_h^{1-\rho}}{1-\rho} + \frac{1}{2} \Sigma_{t=0}^{\infty} \beta^t \frac{c_l^{1-\rho}}{1-\rho} \\ &= \Sigma_{t=0}^{\infty} \beta^t (\frac{1}{2} \frac{c_h^{1-\rho}}{1-\rho} + \frac{1}{2} \frac{c_l^{1-\rho}}{1-\rho}) \end{aligned}$$

(It's still problematic) The intuitive conclusion should be "Early resolution of uncertainty is preferred when risk aversion is high." With EZ preference when $\alpha > \rho$, Lottery A is preferred. when $\alpha < \rho$, Lottery B is preferred. So, we couldn't use lottery A and lottery B to get this conclusion.

Proof:

For Lottery A,
Suppose we have c_h at $t = 1$, we have

$$V_{1,h}^{1-\rho} = (1-\beta) c_h^{1-\rho} + \beta [E(V_2^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}$$

V_2 could be either $V_{2,h}$ or $V_{2,l}$.
Suppose we have c_h at $t = 2$, we have

$$V_{2,h}^{1-\rho} = (1-\beta) c_h^{1-\rho} + \beta [E(V_3^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}$$

V_3 could be either $V_{3,h}$ or $V_{3,l}$.
We could see for $t \geq 1$, every period is the same, so,

$$V_{t,h} \equiv V_h \quad \forall t \geq 1$$

$$V_{t,l} \equiv V_l \quad \forall t \geq 1$$

So

$$V_h^{1-\rho} = (1-\beta) c_h^{1-\rho} + \beta [\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}} \tag{3.3}$$

$$V_l^{1-\rho} = (1-\beta) c_l^{1-\rho} + \beta [\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}} \tag{3.4}$$

$$V_0 = \beta^{\frac{1}{1-\rho}} [\frac{1}{2} V_h^{1-\alpha} + \frac{1}{2} V_l^{1-\alpha}]^{\frac{1}{1-\alpha}} \tag{3.5}$$

For Lottery B, Suppose we have c_h at $t = 1$. Then we have c_h for all $t \geq 1$.
At $t = 1$, we have

$$V_{1,h}^{1-\rho} = (1-\beta) c_h^{1-\rho} + \beta [E(V_{2,h}^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}$$

$V_{2,h}$ is certain, so,

$$V_{1,h}^{1-\rho} = (1-\beta)c_h^{1-\rho} + \beta[V_{2,h}^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}}$$

$$V_{1,h}^{1-\rho} = (1-\beta)c_h^{1-\rho} + \beta[V_{2,h}^{1-\rho}]$$

At $t = 2$, we have

$$V_{2,h}^{1-\rho} = (1-\beta)c_h^{1-\rho} + \beta[V_{3,h}^{1-\rho}]$$

which implies,

$$V_{1,h} = V_{2,h} = V_{3,h} \equiv V_h$$

So, we have,

$$V_h^{1-\rho} = (1-\beta)c_h^{1-\rho} + \beta[V_h^{1-\rho}]$$

which implies, $V_h = c_h$. Similarly, we have $V_{t,l} = V_l = c_l$, all $t \geq 1$. So, at $t = 0$, we have,

$$V_0 = \beta^{\frac{1}{1-\rho}}[\frac{1}{2}c_h^{1-\alpha} + \frac{1}{2}c_l^{1-\alpha}]^{\frac{1}{1-\alpha}}$$

Combining (3.3) and (3.4), notice when $\alpha > \rho$, $\frac{1-\alpha}{1-\rho} < 1$, the function $x^{\frac{1-\alpha}{1-\rho}}$ is concave.

$$\begin{aligned}
&V_h^{1-\alpha} + V_l^{1-\alpha} \\
&= \{(1-\beta)c_h^{1-\rho} + \beta[\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1-\alpha}{1-\rho}} + \{(1-\beta)c_l^{1-\rho} + \beta[\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1-\alpha}{1-\rho}} \\
&> \{(1-\beta)c_h^{1-\alpha} + \beta[\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]\} + \{(1-\beta)c_l^{1-\alpha} + \beta[\frac{1}{2}(V_h^{1-\alpha}) + \frac{1}{2}(V_l^{1-\alpha})]\} \\
&= (1-\beta)(c_h^{1-\alpha} + c_l^{1-\alpha}) + \beta[(V_h^{1-\alpha}) + (V_l^{1-\alpha})]
\end{aligned}$$

So

$$V_h^{1-\alpha} + V_l^{1-\alpha} \geq c_h^{1-\alpha} + c_l^{1-\alpha}$$

From Previous Solutions (incomplete)

Notice that the certainty equivalent $G^{-1}(\cdot)$ can be derived as

$$G(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad \Longleftrightarrow \quad G^{-1}(x) = [(1-\gamma)x]^{\frac{1}{1-\gamma}}$$

and therefore,

$$R_t(V_{t+1}) = G^{-1}\left[\mathbb{E}_t G(V_{t+1})\right] = \left[\mathbb{E}_t(V_{t+1}^{1-\gamma})\right]^{\frac{1}{1-\gamma}}$$

which makes the sequence of lifetime utilities $\{V_t\}_{t=0}^{T}$ defined recursively by

$$V_t = \left\{(1-\beta)c_t^{1-\rho} + \beta\left[\mathbb{E}_t(V_{t+1}^{1-\gamma})\right]^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}}, \quad t = 1, \ldots, T \tag{3.6}$$

and $V_t = 0$ for $t \geq T+1$. Take *lottery A*. Starting from date 1 onward, the consumption is non-stochastic. Since $c_1 = c_2 = \ldots = c_T = C_s$, where $s \in \{h, l\}$, the consumer's lifetime utility can be inductively found as

$$V_T = [1-\beta]^{\frac{1}{1-\rho}} C_s$$

$$V_{T-1} = \left\{ (1-\beta) C_s^{1-\rho} + \beta \left[(1-\beta)^{\frac{1-\gamma}{1-\rho}} C_s^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} = [1-\beta+\beta(1-\beta)]^{\frac{1}{1-\rho}} C_s$$

$$\vdots$$

$$V_1 = \left[(1-\beta)(1+\beta+\ldots+\beta^{T-1}) \right]^{\frac{1}{1-\rho}} C_s = \left[1-\beta^T \right]^{\frac{1}{1-\rho}} C_s$$

while at time 0, *prior to the resolution of uncertainty*, we have (Suppose that at time 0, the consumer does not consume ($c_0 = 0$) and for simplicity, does not discount consumption at the next date.)

$$V_0 = \left\{ \left[\mathbb{E}_0 (V_1^{1-\gamma}) \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} = \left[1-\beta^T \right]^{\frac{1}{1-\rho}} \left[\frac{1}{2} C_l^{1-\gamma} + \frac{1}{2} C_h^{1-\gamma} \right]^{\frac{1}{1-\gamma}}$$

As can be seen, V_0 depends on the coefficient of relative risk aversion, γ. Furthermore, as $T \to \infty$, V_0 no longer depends on the intertemporal elasticity of substitution, $1/\rho$. This is unsurprising, since starting from date 1 onward, consumption does not fluctuate. The fact that it *does* depend on ρ for the finite-horizon case ($T < \infty$) is an artifact of the normalization we use (i.e. that c_t in the brackets has the weight $1-\beta$).

The computation of the expected utility for *lottery C* is quite messy. In general, there can be 2^T different outcomes (that is, possible consumption sequences), the probability of each being $(1/2)^T$.

Indexing consumption by t, for a given realized consumption stream $\{\tilde{c}_1, \ldots, \tilde{c}_T\}$ we have

$$V_T = [1-\beta]^{\frac{1}{1-\rho}} \tilde{c}_T$$

$$\vdots$$

$$V_1 = \left[(1-\beta)(\tilde{c}_1^{1-\rho} + \beta \tilde{c}_2^{1-\rho} + \ldots + \beta^{T-1} \tilde{c}_T^{1-\rho}) \right]^{\frac{1}{1-\rho}} = \left[(1-\beta) \sum_{t=1}^{T} \beta^{t-1} \tilde{c}_t^{1-\rho} \right]^{\frac{1}{1-\rho}}$$

Now, for each $t = 1, \ldots, T$, we have $\tilde{c}_t \in \{C_l, C_h\}$. Denoting by $\times\tilde{c}$ to be the set of all possible consumption paths (there will be 2^T of them in total) with generic element $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_T)$, we have

$$V_0 = \frac{1}{2^T} \left\{ \sum_{\times\tilde{c}} \left[(1-\beta) \sum_{t=1}^{T} \beta^{t-1} \tilde{c}_t^{1-\rho} \right]^{\frac{1-\gamma}{1-\rho}} \right\}^{\frac{1}{1-\gamma}}$$

which depends non-trivially on both ρ and γ. The reader is advised to take the horizon $T = 2$ and compute the value of these two lotteries. Then, one can compare the value of the lottery C with the following *lottery B*: each period, flip a coin; if the flip is head, consume C_h, otherwise consume C_l. Which lottery would the consumer prefer? These two lotteries differ with respect to the *timing of the resolution of uncertainty.* (In *lottery C*, all uncertainty is resolved at $t = 0$). It can be demonstrated that the consumer would prefer early (late) resolution of uncertainty, provided that $\alpha > (<)\rho$.

4 Epstein and Zin Preference and Equity Premium Puzzle

a. We prove for the stochastic discount factor. Let $V_t \equiv F(c_t, R_t(V_{t+1}))$ S is the intertemporal marginal rate of substitution, defined as:

$$S_{t,t+1}(s^t, s_{t+1}) = \frac{\partial V_t/\partial c_{t+1}}{\partial V_t/\partial c_t}\frac{1}{\pi(s^{t+1}|s^t)}$$

Take first order differentials of $V_t = F(c_t, R_t(V_{t+1}))$ w.r.t c_t and c_{t+1}, we have:

$$\frac{\partial V_t}{\partial c_t} = F_1(c_t, R_t(V_{t+1}))$$

$$\frac{\partial V_t}{\partial c_{t+1}} = F_2(c_t, R_t(V_{t+1}))\left[\mathbb{E}_t V_{t+1}^{1-\alpha}\right]^{\frac{\alpha}{1-\alpha}} \pi(s^{t+1}|s^t) V_{t+1}^{-\alpha} F_1(c_{t+1}, R_{t+1}(V_{t+2}))$$

Since

$$\frac{\partial R_t(V_{t+1})}{\partial V_{t+1}} = \left[\mathbb{E}_t V_{t+1}^{1-\alpha}\right]^{\frac{\alpha}{1-\alpha}} V_{t+1}^{-\alpha}$$

From the specification of F and G, we have:

$$F_1(c_t, R_t(V_{t+1})) = (1-\beta)c_t^{-\rho} F(c_t, R_t(V_{t+1}))^{\rho}$$

and

$$F_2(c_t, R_t(V_{t+1})) = \beta R_t(V_{t+1})^{-\rho} V_t^{\rho}$$

From these, we have:

$$\begin{aligned}
S_{t,t+1}(s^t, s_{t+1}) &= \frac{F_2(c_t, R_t(V_{t+1}))\left[\mathbb{E}_t V_{t+1}^{1-\alpha}\right]^{\frac{\alpha}{1-\alpha}} V_{t+1}^{-\alpha} F_1(c_{t+1}, R_{t+1}(V_{t+2}))}{F_1(c_t, R_t(V_{t+1}))} \\
&= \frac{\beta R_t(V_{t+1})^{-\rho} V_t^{\rho}\left[\mathbb{E}_t V_{t+1}^{1-\alpha}\right]^{\frac{\alpha}{1-\alpha}} V_{t+1}^{-\alpha} c_{t+1}^{-\rho} F(c_{t_1}, R_{t+1}(V_{t+2}))^{\rho}}{c_t^{-\rho} V_t^{\rho}} \\
&= \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left[R_t(V_{t+1})^{-\rho}\left[\mathbb{E}_t V_{t+1}^{1-\alpha}\right]^{\frac{\alpha}{1-\alpha}} V_{t+1}^{-\alpha} F(c_{t_1}, R_{t+1}(V_{t+2}))^{\rho}\right] \\
&= \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left[R_t(V_{t+1})^{-\rho} R_t(V_{t+1})^{\alpha} V_{t+1}^{-\alpha} V_{t+1}^{\rho}\right] \\
&= \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{V_{t+1}}{R_t(V_{t+1})}\right)^{\rho-\alpha} \qquad (4.1)
\end{aligned}$$

$$S_{t,t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{V_{t+1}}{R_t(V_{t+1})}\right)^{\rho-\alpha} \qquad (4.2)$$

b. V_t is homogenous of degree one, Eulers theorem implies

$$V_t = MC_t C_t + E_t MV_{t+1} V_{t+1}$$

$$\frac{V_t}{MC_t} = C_t + E_t(\frac{MV_{t+1}MC_{t+1}}{MC_t})\frac{V_{t+1}}{MC_{t+1}}$$

$$\frac{V_t}{MC_t} = C_t + E_t S_{t,t+1}\frac{V_{t+1}}{MC_{t+1}}$$

Which gives us

$$W_t = \frac{V_t}{MC_t}$$

c.

$$W_{t+1} = \frac{V_{t+1}}{MC_{t+1}} = \frac{V_{t+1}^{1-\rho}C_{t+1}^{\rho}}{1-\beta}$$

$$R_{m,t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-\rho} = \left(\frac{V_{t+1}^{1-\rho}}{V_t^{1-\rho} - (1-\beta)C_t^{1-\rho}}\right)$$

Using

$$V_t^{1-\rho} = (1-\beta)C_t^{1-\rho} + \beta R_t(V_{t+1})^{1-\rho}$$

We get

$$R_{m,t+1}^{-1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\rho}\left(\frac{V_{t+1}}{R_t(V_{t+1})}\right)^{\rho-1} \tag{4.3}$$

d. Combining (4.2) and (4.3), we get

$$S_{t,t+1} = \beta^{\theta} R_{m,t+1}^{\theta-1}\left(\frac{c_{t+1}}{c_t}\right)^{-\frac{\theta}{\phi}}$$

Taking logs,

$$logS_{t,t+1} = \theta log\beta - (1-\theta)r_{m,t+1} - \frac{\theta}{\phi}(\Delta c_{t+1})$$

Using the fact that

$$E_t S_{t,t+1} R_{t+1}^{f} = 1$$

As done in the lecture, we have

$$\frac{E_t R_{i,t+1}}{R_{f,t+1}} = -cov(logS_{t,t+1}, logR_{i,t+1}) = \frac{\theta}{\phi}\sigma_{ic} + (1-\theta)\sigma_{im}$$

So the risk premium for an asset i is,

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \frac{\theta}{\phi}\sigma_{ic} + (1-\theta)\sigma_{im}$$

e. Combing $\sigma_{mc} = \sigma_m^2 + (1-\phi)\sigma_{mh}$ and $E_t r_{m,t+1} - r_{f,t+1} + \frac{\sigma_m^2}{2} = \frac{\theta}{\phi}\sigma_{mc} + (1-\theta)\sigma_m^2$, and plugging θ and ϕ, we could get

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_m^2}{2} = \alpha\sigma_m^2 + (1-\alpha)\sigma_{mh}$$

$$\alpha = \frac{0.06}{0.17^2} = 2.07$$

If there is mean reversion and future returns are negatively correlated with current returns then $\sigma_{mh} < 0$ we would need a higher α.

Within the standard RBC framework, equity risk premium is equal to the negative of the covariance between the stochastic discount factor and the returns. The findings of Mehra and Prescott (1985) reveal that in order to match observed returns on equity, one would need to take implausibly low coefficient of the elasticity of substitution. The non-expected utility framework enables to disentangle it from the risk aversion parameter. Some steps toward the resolution of the "Equity Premium Puzzle" were taken by Weil (1989). His results, however, were not very encouraging.

Epstein and Zin consider a class of utility functions that separate risk aversion (to consumption variation at a point in time) from risk aversion to consumption variation across time. They argue that individuals are much more risk averse when it comes to the latter and claim that this phenomenon explain the larger equity risk premiums. Put in more intuitive terms, individuals will choose a lower and more stable level of wealth and consumption that they can sustain over the long term over a higher level of wealth and consumption that varies widely from period to period.

5 Short Questions on Asset Pricing

There is no solutions from the professor. Here are the TA's solutions.

References

Barro, R. J. and R. G. King (1984): "Time-Separable Preferences and Intertemporal-Substitution Models of Business Cycles," *Quarterly Journal of Economics*, 99, 817–839.

Greenwood, J., Z. Hercowitz, and G. W. Huffman (1988): "Investment, Capacity Utilization, and the Real Business Cycle," *American Economic Review*, 78, 402–417.

Intriligator, M. D. (2002): *Mathematical Optimization and Economic Theory*, Society for Industrial and Applied Mathematics, 1 ed.

Kydland, F. E. and E. C. Prescott (1982): "Time to Build and Aggregate Fluctuations," *Econometrica*, 50, 1345–1370.

Ljungqvist, L. and T. J. Sargent (2004): *Recursive Macroeconomic Theory*, MIT Press, 2 ed.

Long, J. B. and C. I. Plosser (1983): "Real Business Cycles," *Journal of Political Economy*, 91, 39–69.

Mehra, R. and E. C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161.

Simon, C. P. and L. E. Blume (1994): *Mathematics for Economists*, W. W. Norton & Company, 1 ed.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.

Weil, P. (1989): "The Equity Premium Puzzle and the Risk-free Rate Puzzle," *Journal of Monetary Economics*, 24, 401–421.

$$\tilde{R}_t^j = R(z^t) + e_t^j$$

$$Cov(e_t^j, m(z^t)) = 0$$

$$Cov(\tilde{R}_t^j, m(z^t))$$

$$= Cov(R(z^t) + e_t^j, m(z^t))$$

$$= Cov(R(z^t), m(z^t)) + \underbrace{Cov(e_t^j, m(z^t))}_{=0}$$

$$= Cov(R(z^t), m(z^t))$$

From (Asset Pricing)

$$1 = E[m(z^t)\tilde{R}_t^j]$$

$$= Cov(m(z^t), \tilde{R}_t^j) + E(m(z^t))E(\tilde{R}_t^j)$$

$$= Cov(R(z^t), m(z^t)) + E(m(z^t))E(\tilde{R}_t^j)$$

since $\frac{1}{R_F} = E(m(z^t))$

$$1 = Cov(R(z^t), m(z^t)) + \frac{E(\tilde{R}_t^j)}{R_F}$$

$$\frac{E(\tilde{R}_t^j) - R_F}{R_F} = -\mathrm{cov}(R(z^t), m(z^t)) \leq \sigma(R(z^t))\,\sigma(m(z^t))$$

$$\frac{E(\tilde{R}_t^j) - R_F}{\sigma(R(z^t))} \leq R_F\,\sigma(m(z^t))$$

$$\frac{E(\tilde{R}_t^j) - R_F}{\sigma(\tilde{R}_t^j)} \underbrace{\frac{\sigma(\tilde{R}_t^j)}{\sigma(R(z^t))}} \leq R_F\,\sigma(m(z^t))$$

$$\swarrow$$

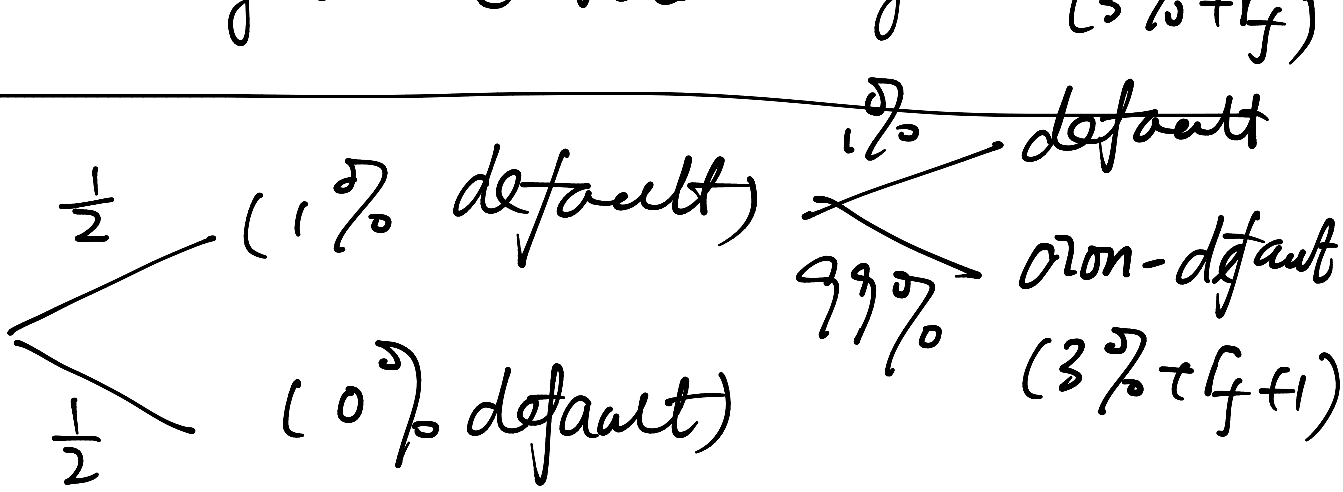
$$\frac{\sqrt{\sigma^2(R(z^t)) + \sigma^2(e_t^j)}}{\sigma(R(z^t))} > 1$$

original $\dfrac{E(\tilde{R}_t^j) - R_F}{\sigma(\tilde{R}_t^j)} \leq R_F\,\sigma(m(z^t))$

a larger lower bound than the original one!

For securities that $\sigma(e_i^j) \neq 0$, this is important.

i.e. securities with significant idiosyncratic volatility.



$$E(\tilde{R}) = \frac{1}{2} \times 1\% \times (3\% + r_f) + \frac{1}{2} \times 99\% \times (3\% + r_f + 1) + \frac{1}{2}(3\% + r_f + 1)$$

$$= 3\% + r_f + 99.5\%$$

$$Var(\tilde{R}) = 0.5\% \times (99.5\%)^2 + 99.5\% \times (0.5\%)^2$$

$$= 0.5\% \times 99.5\%$$

$$\sigma(\tilde{R}) = 7.05\%$$

$$\text{sharpe ratio} = \frac{E(\tilde{R}) - (1 + r_f)}{\sqrt{Var(\tilde{R})}} = \frac{2.5\%}{7.05\%}$$

$$\approx 0.354$$