Macroeconomics I—Problem Set 8– Solutions

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TAs 2013: George Lukyanov, Harry Di Pei

TAs 2014: Constance de Soyres and Francois de Soyres

TAs 2015: Yue Fei and Lan lan

1 A simple RBC Model

At time t, representative consumer chooses the sequences $\{C_{t+j}, N_{t+j}, \widetilde{K}_{t+j+1}\}_{j=0}^{\infty}$, solving

$$\max_{\{C_{t+j}, N_{t+j}, \widetilde{K}_{t+j+1}\}} \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \left(\log \widetilde{C}_{t+j} + \phi \log(1 - N_{t+j}) \right) \right\} \tag{1.1}$$

subject to

$$\widetilde{K}_{t+j+1} = R_t \widetilde{K}_{t+j} + \widetilde{W}_{t+j} N_{t+j} - \widetilde{C}_{t+j} \tag{1.2}$$

while the representative firm each period chooses N_t and $\widetilde{K}_t$, solving[1]

$$\max_{N_t, \widetilde{K}_t} \left\{ Z_t \widetilde{K}_t^{1-\alpha} (A_t N_t)^{\alpha} - (R_t + \delta - 1) \widetilde{K}_t - \widetilde{W}_t N_t \right\} \tag{1.3}$$

and we assume that labor productivity process $\{A_t\}_{t=0}^{\infty}$ evolves deterministically, growing at a (gross) constant rate of γ: $A_{t+1} = \gamma A_t$.

1. Setting up the Lagrangian for consumer's problem, we get

   $$\mathcal{L} =$$

   $$\mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \mathbb{E}_{t+j} \left[\beta^j \left(\log \widetilde{C}_{t+j} + \phi \log(1 - N_{t+j}) \right) \right] + \lambda_j \left[R_t \widetilde{K}_{t+j} + \widetilde{W}_{t+j} N_{t+j} - \widetilde{C}_{t+j} - \widetilde{K}_{t+j+1} \right] \right\}$$

   where we have used the Law of Iterated Expectations. FOC's with respect to C_{t+j}, N_{t+j} and K_{t+j+1} yield

   $$\{C_{t+j}\} : \quad \frac{\beta^j}{C_{t+j}} = \lambda_j \tag{1.4}$$

   $$\{N_{t+j}\} : \quad \frac{\beta^j \phi}{1 - N_{t+j}} = \lambda_j \widetilde{W}_{t+j} \tag{1.5}$$

   $$\{K_{t+j+1}\} : \quad \lambda_j = \mathbb{E}_{t+j} \left[\lambda_{j+1} R_{t+j+1} \right] \tag{1.6}$$

[1] Notice that there is no intertemporal link in the firm's objective – hence, we can represent it as a sequence of *static*, period-by-period maximization problems.

Substituting λ_j from (1.4) into (1.6) and simplifying, we obtain the Euler equation:

$$1 = \beta \mathbb{E}_{t+j} \left[\frac{C_{t+j}}{C_{t+j+1}} R_{t+j+1} \right] \tag{1.7}$$

Now, substituting for λ_j from (1.4) into (1.5), we get the labor supply:

$$N_{t+j} = 1 - \phi \frac{C_{t+j}}{\widetilde{W}_{t+j}} \tag{1.8}$$

2. Taking FOC's with respect to N_t and $\widetilde{K}_t$ in (1.3), we get

N_t :
$$\alpha Z_t \widetilde{K}_t^{1-\alpha} A_t^{\alpha} N_t^{\alpha-1} = \widetilde{W}_t \tag{1.9}$$

$\widetilde{K}_t$:
$$(1-\alpha) Z_t \widetilde{K}_t^{-\alpha} (A_t N_t)^{\alpha} = R_t + \delta - 1 \tag{1.10}$$

Using the fact that $Z_t \widetilde{K}_t^{1-\alpha} (A_t N_t)^{\alpha} \equiv Y_t$, rewrite firms' labor and capital demands as

$$N_t = \alpha \frac{Y_t}{\widetilde{W}_t}$$

and

$$\widetilde{K}_t = (1-\alpha) \frac{Y_t}{R_t + \delta - 1}$$

Since the production function is characterized by constant returns to scale, all the output is used for payments to capital and labor. As can be seen from the above two equations,

$$Y_t = \widetilde{W}_t N_t + (R_t + \delta - 1) \widetilde{K}_t \Longleftrightarrow \widetilde{W}_t N_t + R_t \widetilde{K}_t = Y_t + (1-\delta) \widetilde{K}_t$$

and so, the law of motion for capital is given by

$$\widetilde{K}_{t+1} = Y_t - C_t + (1-\delta) \widetilde{K}_t \tag{1.11}$$